PROBABILITY I, SPRING 2017, HW1

BOB HOUGH

Problem 1. A set $A \subset \{1, 2, ...\}$ is said to have *asymptotic density* θ if

$$\lim_{n\to\infty} \frac{1}{n}|A \cap \{1, 2, ..., n\}| = \theta.$$

Let $\mathscr{A}$ be the collection of sets for which the asymptotic density exists. Is $\mathscr{A}$ a σ-algebra? An algebra?

Problem 2. Let $Y \geqslant 0$ with $\mathbf{E}[Y^2] < \infty$. Prove $\mathrm{Prob}(Y > 0) \geqslant \frac{\mathbf{E}[Y]^2}{\mathbf{E}[Y^2]}$.

Problem 3. When two standard dice are rolled, the probability distribution of the sum is distributed as

$$p(n) = \frac{6 - |7 - n|}{36}, \qquad 2 \leqslant n \leqslant 12.$$

Find two six-sided dice A and B with positive numbers other than 1–6, which, when rolled, give the same probability distribution for the sum.

Problem 4. Show that if X and Y are independent with distributions μ and ν, then

$$\mathrm{Prob}(X + Y = 0) = \sum_y \mu(\{-y\})\nu(\{y\}).$$

Conclude that if X has continuous distribution function, then $\mathrm{Prob}(X = Y) = 0$.

Problem 5. Let $X \geqslant 0$. Show

$$\lim_{y\to\infty} y\, \mathbf{E}\left[X^{-1} \cdot \mathbf{1}_{(X>y)}\right] = 0, \qquad \lim_{y\downarrow 0} y\, \mathbf{E}\left[X^{-1} \cdot \mathbf{1}_{(X>y)}\right] = 0.$$