Problem 1. Let $a = (a_1, \ldots, a_n)$ be a vector whose coordinates are linearly independent over $\mathbb{Q}$. Prove that any Borel measurable set $E$ of $(\mathbb{R}/\mathbb{Z})^n$ which is invariant under translation by $a$ has measure 0 or 1.

Problem 2. Let $C : \mathcal{S}(\mathbb{R}^n) \to C(\mathbb{R}^n)$ be a continuous linear map which commutes with translation. Prove that $C$ is given by convolution with a tempered distribution.

Problem 3. Form an $n \times n$ matrix by giving its entries independent mean 0 variance 1 standard Gaussian variables, then performing the Gram-Schmidt process on its columns treated as vectors. Prove that with probability 1 the resulting matrix is orthogonal, and that the distribution thus obtained is Haar measure on the orthogonal group.

Problem 4. A bi-infinite sequence $\{a_n\}_{n \in \mathbb{Z}}$ is positive definite if, for all finite sets of complex numbers $\phi_n, -N \leq n \leq N$, we have

$$\sum_{n,k} a_{n-k} \phi_n \overline{\phi_k} \geq 0.$$ 

Prove that a sequence $\{a_n\}_{n \in \mathbb{Z}}$ is positive definite if and only if it is the set of Fourier coefficients of a positive measure on $\mathbb{R}/\mathbb{Z}$. (Hint: first show that these are the Fourier coefficients of a distribution.)

Problem 5. Let $\Omega \subset \mathbb{C}$ be an open domain, and let $f : \Omega \to X$ be a map to a complex Banach space $X$. The function $f$ is said to be strongly analytic if the difference quotients

$$\lim_{k \to 0} \frac{1}{k} (f(x + k) - f(x))$$

exist at each point. The function $f$ is said to be weakly analytic if, for each bounded linear functional $\ell$, $\ell(f(x))$ is an analytic function in the usual sense. Prove that $f$ is strongly analytic if and only if it is weakly analytic. It may help to use the contour formula

$$\frac{1}{h - k} \left[ \frac{\ell(f(z + h))}{h} - \frac{\ell(f(z + k))}{k} \right] = \frac{1}{2\pi i} \int_C \ell(f(\zeta)) \frac{d\zeta}{(\zeta - z - h)(\zeta - z - k)(\zeta - z)}$$

where $C$ is a smooth contour with winding number 1 about $z, z + h, z + k$. 

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