Problem 1. A distribution $F$ is called harmonic if $\sum_1^n \partial_j^2 F = 0$. If $F$ is harmonic and tempered, prove that $F$ is a polynomial.

Problem 2. Let $f$ be a continuous function on $\mathbb{R}^n \setminus \{0\}$ which is homogeneous of degree $-n$, $f(rx) = r^{-n}f(x)$ for $r > 0$, and has mean 0 on the unit sphere. Show that $f$ is not locally integrable near 0 unless $f = 0$, but $f$ defines a tempered distribution $PV(f)$ by

$$\langle PV(f), \phi \rangle = \lim_{\epsilon \to 0} \int_{|x| > \epsilon} f(x)\phi(x)dx.$$ 

The limit equals

$$\int_{|x| \leq 1} f(x)[\phi(x) - \phi(0)]dx + \int_{|x| > 1} f(x)\phi(x)dx,$$

and these integrals converge absolutely.

Problem 3. On $\mathbb{R}$, let $F = PV((\pi x)^{-1})$. Check that

1. $\hat{F}(\xi) = -i \text{sgn } \xi$.
2. The map $\phi \to F * \phi$, initially defined on Schwartz functions, extends to a unitary operator on $L^2$.

This is called the Hilbert transform.

Problem 4. Give $C^\infty(\mathbb{T}^n)$ the Fréchet space topology defined by the semi-norms $\|\phi\|_{(\alpha)} = \|\partial^\alpha \phi\|_{\infty}$. The space $\mathcal{D}'(\mathbb{T}^n)$ of distributions on $\mathbb{T}^n$ is the space of continuous linear functionals on $C^\infty(\mathbb{T}^n)$, with the weak * topology. Prove the following.

1. Distributions on $\mathbb{T}^n$ can be translated, differentiated, and multiplied by $C^\infty$ functions, just as on $\mathbb{R}^n$. 

(2) If $F \in \mathcal{D}'(\mathbb{T}^n)$, its Fourier transform is the function $\hat{F}$ on $\mathbb{Z}^n$ defined by $\hat{F}(\kappa) = \langle F, E_\kappa \rangle$ where $E_\kappa(x) = e^{-2\pi i \kappa \cdot x}$. Prove that a function $g$ on $\mathbb{Z}^n$ is the Fourier transform of a distribution on $\mathbb{T}^n$ iff $|g(\kappa)| \leq C(1 + |\kappa|)^N$ for some $C, N > 0$.

(3) If $F \in \mathcal{D}'(\mathbb{T}^n)$ and $\phi \in C^\infty(\mathbb{T}^n)$, prove $\langle F, \phi \rangle = \sum_\kappa \hat{F}(\kappa) \hat{\phi}(\kappa)$.

Problem 5. If $k \in \mathbb{N}$, $H_k$ is the space of $L^2$ functions whose $L^2$ derivatives up to order $k$ exist. Show that these strong derivatives coincide with the distribution derivatives.