Problem 1. Suppose \( f \in L^p(\mathbb{R}) \). If there exists \( h \in L^p(\mathbb{R}) \) such that
\[
\lim_{y \to 0} \|y^{-1}(f^{-y} - f) - h\|_p = 0,
\]
we call \( h \) the \textit{strong} \( L^p \) derivative of \( f \) and write \( h = df/dx \). If \( f \in L^p(\mathbb{R}^n) \), \( L^p \) derivatives of \( f \) are defined similarly. If \( p \) and \( q \) are conjugate exponents, \( f \in L^p \), \( g \in L^q \), and the \( L^p \) derivative \( \partial_j f \) exists, then prove \( \partial_j (f \ast g) \) exists in the ordinary sense and equals \((\partial_j f) \ast g\).

Problem 2. Let \( \phi \in L^1(\mathbb{R}^n) \) satisfy \(|\phi(x)| \leq C (1 + |x|)^{-n-\epsilon} \) for some \( C, \epsilon > 0 \), and \( \int \phi(x) dx = a \). For \( t > 0 \), \( \phi_t(x) = t^{-n}\phi(\frac{x}{t}) \). If \( f \in L^p \) define the \( \phi \)-maximal function of \( f \) to be \( M_\phi f(x) = \sup_{t>0} |f \ast \phi_t(x)| \). The \textit{Hardy-Littlewood maximal function} \( Hf \) is \( M_\phi|f| \) where \( \phi \) is the characteristic function of the unit ball, divided by the volume of the ball. Show that there is a constant \( C \), independent of \( f \), such that \( M_\phi f \leq CHf \).

Problem 3. Young’s inequality shows that \( L^1 \) is a Banach algebra with convolution as multiplication.

1. If \( I \) is an ideal in the algebra \( L^1 \), prove that its closure is, also.
2. If \( f \in L^1 \), the smallest closed ideal in \( L^1 \) containing \( f \) is the smallest closed subspace of \( L^1 \) containing translates of \( f \).

Problem 4. Show that if \( f \in L^1(\mathbb{R}^n) \), \( f \) is continuous at 0, and \( \hat{f} \geq 0 \), then \( \hat{f} \in L^1 \).

Problem 5. Let \( f \) be a function on \( \mathbb{T}^1 \) and \( A_r f \) the \( r \)th Abel mean of the Fourier series of \( f \). Check that

1. \( A_r f = f \ast P_r \) where \( P_r(x) = \sum_{-\infty}^{\infty} r^{|k|} e^{2\pi i k x} \) is the Poisson kernel for \( \mathbb{T}^1 \).
2. \( P_r(x) = \frac{1 - r^2}{1 + r^2 - 2r \cos 2\pi x} \).
Problem 6. Given $f \in L^1(T^1)$, let $S_m f(x) = \sum_{-m}^{m} \hat{f}(k)e^{2\pi ikx}$ and 

$$\sigma_m f(x) = \sum_{-m}^{m} \hat{f}(k) \left(1 - \frac{|k|}{m+1}\right)e^{2\pi ikx}.$$ 

Prove the following.

1. $\sigma_m f = \frac{1}{m+1} \sum_{0}^{m} S_k f$.
2. If $D_k$ is the $k$th Dirichlet kernel, we have $\sigma_m f = f \ast F_m$ where $F_m = \frac{1}{m+1} \sum_{0}^{m} D_k$. $F_m$ is the $m$th Fejér kernel on $T^1$.
3. $F_m(x) = \frac{\sin^2((m+1)\pi x)}{(m+1)\sin^2\pi x}$.

Problem 7. Prove the following.

1. If $D_m$ is the $m$th Dirichlet kernel, $\|D_m\|_1 \to \infty$ as $m \to \infty$.
2. The Fourier transform is not surjective from $L^1(T^1)$ to $C_0(\mathbb{Z})$. 
