Problem 1. Let $K \in C([0, 1] \times [0, 1])$. For $f \in C([0, 1])$ define $Tf(x) = \int_0^1 K(x, y)f(y)dy$. Show that $Tf \in C([0, 1])$, and $\{Tf : \|f\|_u \leq 1\}$ is precompact in $C([0, 1])$.

Problem 2. Let $U$ be an open subset of $\mathbb{C}$ and $\{f_n\}$ a sequence of holomorphic functions on $U$. If $\{f_n\}$ is uniformly bounded on compact subsets of $U$, show that there is a subsequence which converges uniformly to a holomorphic function on compact subsets of $U$.

Problem 3. Let $X$ and $Y$ be compact Hausdorff spaces. Show that the algebra generated by functions of the form $f(x, y) = g(x)h(y)$, where $g \in C(X)$ and $h \in C(Y)$, is dense in $C(X \times Y)$.

Problem 4. Let $(X, \mathcal{M})$ be a measurable space, and let $M(X)$ be the space of complex measures on $(X, \mathcal{M})$. Show that then $\|\mu\| = |\mu|(X)$ is a norm on $M(X)$ which makes $M(X)$ into a Banach space.

Problem 5. If $0 < \alpha \leq 1$, let $\Lambda_{\alpha}([0, 1])$ be the set of Hölder continuous functions of exponent $\alpha$ on $[0, 1]$. Thus $f \in \Lambda_{\alpha}([0, 1])$ iff $\|f\|_{(\alpha)} < \infty$, where

$$
\|f\|_{(\alpha)} = |f(0)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.
$$

(1) Show that $\|\cdot\|_{(\alpha)}$ is a norm which makes $\Lambda_{\alpha}([0, 1])$ into a Banach space.

(2) Let $\lambda_{\alpha}([0, 1])$ be the set of all $f \in \Lambda_{\alpha}([0, 1])$ such that

$$
\frac{|f(x) - f(y)|}{|x - y|^\alpha} \to 0, \quad \text{as } x \to y, \text{ for all } y \in [0, 1].
$$

If $\alpha < 1$, show that $\lambda_{\alpha}([0, 1])$ is a closed subspace of $\Lambda_{\alpha}([0, 1])$, while if $\alpha = 1$, $\lambda_{\alpha}([0, 1])$ contains only constant functions.
Problem 6. Show that a linear functional $f$ on a normed vector space $X$ is bounded iff $f^{-1}(\{0\})$ is closed.

Problem 7. If $M$ is a finite-dimensional subspace of a normed vector space $X$, show that there is a closed subspace $N$ such that $M \cap N = \{0\}$ and $M + N = X$.

Problem 8. If $X$ is a Banach space, and $X^*$ is separable, show that $X$ is separable.