Problem 1. If $X_n \to X$ in probability, then $P_{X_n} \to P_X$ vaguely.

Problem 2. Identify $T^1$ with $\{z \in \mathbb{C} : |z| = 1\}$.

1. If $X_1, ... , X_n$ are independent, then $P_{X_1X_2...X_n} = P_{X_1} * \cdots * P_{X_n}$.
2. If $\{X_j\}$ is a sequence of independent random variables with common distribution $\lambda$, the distribution of $\prod^n_1 X_j$ converges vaguely to the uniform distribution on $T^1$ unless $X_1$ is supported on a finite subgroup of $T^1$.

Problem 3. Given $b \in \mathbb{N} \setminus \{1\}$, let $B = \{0, 1, ..., b - 1\}$ and $\Omega = B^\mathbb{N}$. Put the discrete topology on $B$ and the product topology on $\Omega$, and let $P$ be the product measure on $\Omega$, where each $P_n$ is $b^{-1}$ times counting measure on $B$. Let $\{X_n\}^\infty_1$ be the coordinate functions on $\Omega$. Then if $A_1, ..., A_n \subset B$,

$$\text{Prob} \left( \bigcap^n_1 X_j^{-1}(A_j) \right) = b^{-n} \prod^n_1 |A_j|$$

and $P(\{\omega\}) = 0$ for all $\omega \in \Omega$.

Problem 4. Prove the following. Let

$$\Omega' = \{\omega \in \Omega : X_n(\omega) \neq 0 \text{ for infinitely many } n\}.$$ 

Then $\Omega \setminus \Omega'$ is countable and $P(\Omega') = 1$. Define $F : \Omega \to [0, 1]$ by $F(\omega) = \sum^\infty_1 X_n(\omega)b^{-n}$. Then $F|_{\Omega'}$ is a bijection from $\Omega'$ to $(0, 1]$ which maps $B_{\Omega'}$ bijectively onto $B_{(0,1]}$.

Problem 5. (Borel’s normal number theorem) A number $x \in (0, 1]$ is called normal in base $b$ if the digits $0, 1, ... , b - 1$ occur with equal frequency in its base $b$ decimal expansion, that is, if $n^{-1}X_j^{-1}(F^{-1}(x)) \to b^{-1}$ as $n \to \infty$. Almost every $x \in (0, 1]$ (with respect to Lebesgue measure) is normal in base $b$ for every $b$. 

1