Each problem is worth 10 points.
Problem 1.
  a. State and prove Bessel’s inequality for a Hilbert space \( \mathcal{H} \).

b. Using Bessel’s inequality, or otherwise, prove that if \( \mathcal{H} \) has a countable orthonormal basis, then any orthonormal basis of \( \mathcal{H} \) is countable.
Problem 2.

a. Let $\mathcal{X}$ be an infinite dimensional normed vector space. Prove that the unit ball $B_1 = \{ x \in \mathcal{X} : \|x\| \leq 1 \}$ is not compact in the norm topology.

b. Prove Alaoglu’s Theorem: Let $\mathcal{X}$ be a Banach space. Prove that the unit ball in $\mathcal{X}^*$

$$B_1 = \{ \ell \in \mathcal{X}^* : \|\ell\| \leq 1 \}$$

is compact in the weak-* topology. (Hint: identify $B_1$ with a subspace of $\prod_{x \in \mathcal{X}} [-\|x\|, \|x\|].$)
Problem 3. Define the following sequence spaces of sequences of real numbers.

- For \( p \geq 1 \), \( \ell_p = \{a = \{a_n\}_{n=1}^\infty : \|a\|_p = \sum_n |a_n|^p \} \)
- \( \ell_\infty = \{a = \{a_n\}_{n=1}^\infty : \|a\|_\infty = \sup_n |a_n| \} \)
- \( c_0 = \{a = \{a_n\} : \lim_n a_n = 0, \|a\|_\infty = \sup_n |a_n| \}. \)

a. Prove that \( \ell_p \) is separable, but \( \ell_\infty \) is not.

b. Prove \( c_0^* = \ell_1 \), \( \ell_1^* = \ell_\infty \) but \( \ell_\infty^* \neq \ell_1 \) by using Hahn-Banach. Give an example of a sequence in \( \ell_1 \) which does not converge weakly, but converges weak-*.
Problem 4. Let $\phi \in C_c^\infty(\mathbb{R}^n)$, $\int \phi = 1$, and for real $t > 0$, let $\phi_t(x) = t^{-n}\phi\left(\frac{x}{t}\right)$. Let $1 \leq p < \infty$ and let $f \in L^p(\mathbb{R}^n)$. Prove that $\phi_t * f \in C_c^\infty(\mathbb{R}^n)$ and $\phi_t * f \to f$ in $L^p$ as $t \downarrow 0$. 
Problem 5. Let $\mu$ be a Radon measure on $X$. Prove that $\mu$ is inner regular on Borel sets of finite measure.
Problem 6.
a. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $L(\mathcal{X}, \mathcal{Y})$ be the bounded linear maps between $\mathcal{X}$ and $\mathcal{Y}$. Give a neighborhood base at 0 for the strong and weak operator topologies.

b. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $T_n \in L(\mathcal{X}, \mathcal{Y})$ be such that, for each $x \in \mathcal{X}$, \{\{T_n x\}\} is Cauchy. Prove that $T_n$ converges strongly to some $T \in L(\mathcal{X}, \mathcal{Y})$. 