PROBLEM SET 2

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Problem 1.

Proof. Continuity of Tf follows from the uniform continuity of K and the following estimation

$$\begin{aligned}|Tf(x) - Tf(x')| &\le \int_0^1 |K(x,y) - K(x',y)||f(y)|dy \\ &\le \sup_{y\in[0,1]} |K(x,y) - K(x',y)| \cdot \sup_{[0,1]} |f|.\end{aligned}$$

This estimation also proves $\mathcal{F} = \{Tf : \|f\|_u \le 1\}$ is equicontinuous. Note A is pointwise bounded since

$$|Tf(x)| \le \int_0^1 |K(x,y)||f(y)|dy \le \sup_{[0,1]\times[0,1]} |K| \cdot \sup_{[0,1]} |f|.$$

Then by Arzela-Ascoli, $\mathcal{F}$ is precompact. □

Problem 2.

Proof. For any $x \in U$, we may choose r sufficiently small such that $\bar{B}_r(x) \subset U$. Since $\{f_n\}$ is uniformly bounded on compact subsets of U, $\sup_{z\in\bar{B}_r(x),n\in\mathbb{N}} |f_n(z)|$ is finite and $\{f_n(x)\}$ is bounded for all $x \in U$.

Let $\gamma = \partial B_r(x)$ with counterclockwise orientation. Then if $|x - y| < r/2$, by Cauchy integral formula

$$\begin{aligned}|f_n(x) - f_n(y)| &= \frac{1}{2\pi}\left|\int_\gamma \frac{f_n(z)}{z-x} - \frac{f_n(z)}{z-y}dz\right| \\ &\le \frac{1}{2\pi}\int_\gamma \frac{|f_n(z)|\cdot|x-y|}{|z-x|\cdot|z-y|}dz \\ &\le \frac{1}{2\pi}\sup_{z\in\bar{B}_r(x),n\in\mathbb{N}} |f_n(z)|\cdot|x-y|\cdot\int_\gamma \frac{1}{|z-x|}\cdot\frac{1}{|z-x|-|x-y|}dz \\ &\le \frac{2}{r}\sup_{z\in\bar{B}_r(x),n\in\mathbb{N}} |f_n(z)|\cdot|x-y| \quad \text{note } |z-x| = r \text{ and } |x-y| < r/2.\end{aligned}$$

Therefore $\{f_n\}$ is equicontinuous. So by Arzela-Ascoli, f_n has a subsequence that uniformly converges to a function $f \in C(U)$. Replacing $\{f_n\}$ by a subsequence if necessary, we may assume $f_n \to f$ uniformly on compact subsets, then $\int_\gamma f = \int_\gamma \lim_n f_n = \lim_n \int_\gamma f_n = 0$. So by Morera's theorem, f is homolorphic. □

Problem 3.

Proof. It is clear the algebra $\mathcal{A}$ generated by $C(X) \times C(Y)$ contains non-zero constant functions, so by Stone-Weierstrass theorem, it suffices to prove the algebra $\mathcal{A}$ separate points. Let $(x, y) \neq (x', y') \in X \times Y$, we may assume $x \neq x'$. Since X is compact Hausdorff, by Urysohn's lemma we can find $g \in C(X)$ so that $g(x) \neq g(x')$, and let h be a non-zero constant function on Y, then $f(x,y) = g(x)h(y) \in \mathcal{A}$ seperates (x,y) and (x',y'). $\square$

Problem 4.

Proof. To see $\|\mu\| = |\mu|(X)$ is a norm, we note

- $\|\lambda\mu\| = |\lambda\mu|(X) = |\lambda||\mu|(X) = |\lambda|\|\mu\|$ for all $\lambda \in \mathbb{C}, \mu \in M(X)$;
- $\|\mu+\nu\| = |\mu+\nu|(X) \leq |\mu|(X) + |\nu|(X) = \|\mu\| + \|\nu\|$ for all $\mu, \nu \in M(X)$;
- if $\|\mu\| = 0$, then for any $E \in \mathcal{M}$, $|\mu(E)| \leq |\mu|(E) \leq |\mu|(X) = \|\mu\| = 0$, so $\mu = 0$.

To see $(M(X), \|\cdot\|)$ is a Banach space, we show every absolutely convergent sequence $\{\mu_n\}$ is convergent. Now for any $E \in M(X)$, since $\sum_n |\mu_n(E)| = \sum_n |\mu_n|(E) \leq \sum_n \|\mu_n\| < \infty$, $\sum \mu_n(E)$ converges absolutely to some number $\mu(E)$. In particular, $\|\sum_n \mu_n - \mu\| = \lim_{N\to\infty} |\sum_1^N \mu_n - \mu|(X) = 0$. It remains to check μ is a complex measure on X. Indeed,

- $\mu(\varnothing) = \sum_n \mu_n(\varnothing) = \sum_n 0 = 0$
- let $E_m \in \mathcal{M}, m \in \mathbb{N}$ be disjoint, then

$$\begin{aligned}
\mu(\cup_m E_m) &= \sum_n \mu_n(\cup_m E_m) = \sum_n \sum_m \mu_n(E_m) \\
&= \sum_m \sum_n \mu_n(E_m) \quad \text{since } \sum_m \mu_n(E_m) \text{ absolutely converges} \\
&= \sum_m \mu(E_m) \quad \text{and this convergence is absolute.}
\end{aligned}$$

$\square$

Problem 5.

Proof. (1) It is quite standard to check $\|\cdot\|_\alpha$ is a norm, we omit this part. To show this norm is complete, let $\{f_n\}$ be a Cauchy sequence in $\Lambda_\alpha[0,1]$, then

- $|f_n(0) - f_m(0)| \leq \|f_n - f_m\|_\alpha$, hence $\{f_n(0)\}$ is Cauchy and converges.
- for $x \in (0,1]$, $|f_n(x) - f_m(x)| \leq \|f_n - f_m\||x|^\alpha + |f_n(0) - f_m(0)|$, hence $\{f_n(x)\}$ is Cauchy and converges.

Define $f(x) = \lim_n f_n(x)$ for all $x \in [0,1]$. Then

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
&\leq |f(x) - f_n(x)| + \sup_n \|f_n\|_\alpha \cdot |x-y|^\alpha + |f_n(y) - f(y)|
\end{aligned}$$

Taking $n \to \infty$, we have

$$|f(x) - f(y)| \leq \sup_n \|f_n\|_\alpha \cdot |x-y|^\alpha$$

therefore $\|f\|_\alpha \leq |f(0)| + \sup_n \|f_n\|_\alpha < \infty$, this proves $f \in \Lambda_\alpha[0,1]$. Moreover,

$$\begin{aligned}
|f_n(x) - f_n(y) - [f_m(x) - f_m(y)]| &\leq \|f_n - f_m\|_\alpha \cdot |x-y|^\alpha \\
&\leq \varepsilon \cdot |x-y|^\alpha \quad \text{for } n, m > N_\varepsilon
\end{aligned}$$

Taking $m \to \infty$, we have

$$|f_n(x) - f_n(y) - [f(x) - f(y)]| \leq \varepsilon |x-y|^\alpha \quad n > N_\varepsilon$$

so $\|f_n - f\|_\alpha \leq |f_n(0) - f(0)| + \varepsilon$ for n big, this implies $\|f_n - f\|_\alpha \to 0$.

(2) If $f \in \lambda_1$, then it follows from definition of λ_1 that f' exists and is identically zero, so f is a constant function. Now if $0 < \alpha < 1$, let $\{f_n\} \subset \lambda_\alpha[0,1]$ be a convergent sequence and f be its limit, we show $f \in \lambda_\alpha[0,1]$. For any $y \in [0,1]$, we have

$$\begin{aligned}\frac{|f(x)-f(y)|}{|x-y|^\alpha} &\leq \frac{||f-f_n|(x)-|f-f_n|(y)|}{|x-y|^\alpha} + \frac{|f_n(x)-f_n(y)|}{|x-y|^\alpha} \\ &\leq \|f-f_n\|_\alpha + \frac{|f_n(x)-f_n(y)|}{|x-y|^\alpha}\end{aligned}$$

So

$$\lim_{x\to y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq \|f-f_n\|_\alpha \to 0.$$

This proves $f \in \lambda_\alpha[0,1]$, hence $\lambda_\alpha[0,1]$ is closed.

□

Problem 6.

Proof. Boundedness of f implies countinuity of f, thus implies closedness of $f^{-1}(\{0\})$. Conversely if $f^{-1}(\{0\})$ is closed, we show f is bounded. Otherwise one can find a sequence $\{x_n\}$ in $\mathcal{X}$ with $\|x_n\| = 1$ and $0 < \lambda_n = f(x_n) \to \infty$. Moreover since f cannot be identically zero, we can find $y \in \mathcal{X}\backslash f^{-1}(\{0\})$ with $f(y) = 1$. Now consider the sequence $\{y_n = x_n/\lambda_n - y\}$. Notice that $f(y_n) = f(x_n)/\lambda_n - f(y) = 0$, so $\{y_n\} \subset f^{-1}(\{0\})$. Meanwhile $\|y_n - y\| = \|x_n/\lambda_n\| = 1/\lambda_n \to 0$, so y_n converges to y. By closedness, $y \in f^{-1}(\{0\})$, but this contradicts $f(y) = 1$. □

Problem 7.

Proof. Consider the partial ordered set

$$\mathcal{P} = \{N \text{ closed subspace of } \mathcal{X} : N \cap \mathcal{M} = 0\}$$

whose partial order is the inclusion. $\mathcal{P}$ is non-empty since $\{0\} \in \mathcal{P}$, then by Zorn's lemma $\mathcal{P}$ has a maximal element $\mathcal{N}$. We claim that $\mathcal{M} + \mathcal{N} = \mathcal{X}$. Otherwise, one can find a non-zero vector $x \in \mathcal{X}\backslash(\mathcal{M}+\mathcal{N})$. Define $\mathcal{N}_0 = \mathcal{N} + \mathbb{C}x$, then $\mathcal{N}_0 \cap \mathcal{M} = 0$ and $\mathcal{N} \subsetneqq \mathcal{N}_0$.

Moreover, $\mathcal{N}_0$ is closed. To see this, let $x_n = y_n + \lambda_n x \in \mathcal{N}_0$ be a convergent sequence with limit $z \in \mathcal{X}$, where $y_n \in \mathcal{N}, \lambda_n \in \mathbb{C}$ are uniquely determined by x_n. By Hahn-Banach theorem, there is a countinuous function f such that $f|_{\mathcal{N}} = 0$ and $f(x) = 1$, then $f(x_n) = \lambda_n$ converges to $f(z) =: \lambda$. Therefore $y_n = x_n - \lambda_n x \in \mathcal{N}$ converges to $z - \lambda x$. Thus by closedness of $\mathcal{N}$, $z - \lambda x \in \mathcal{N}$, hence $z = (z-\lambda x) + \lambda x \in \mathcal{N}_0$. So $\mathcal{N}_0$ is closed, thus $\mathcal{N}_0 \in \mathcal{P}$, but this contradicts maximality of $\mathcal{N}$. □

Remark. *Notice that we just showed: if $\mathcal{N}$ is a closed subspace of a normed vector space $\mathcal{X}$ and $x \in \mathcal{X}\backslash\mathcal{N}$, then $\mathcal{N} + \mathbb{C}x$ is closed. Then by induction one can prove if $\mathcal{M}$ is finite dimensional and $\mathcal{M} \cap \mathcal{N} = 0$ then $\mathcal{M} + \mathcal{N}$ is closed. In particular,* every finite dimensional subspace of a normed vector space is closed.

Problem 8.

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a countable dense subset in $\mathcal{X}^*$. For each n choose $x_n \in \mathcal{X}$ with $\|x_n\| = 1$ such that $f_n(x_n) \geq \|f_n\|/2$. We show the countable set of finite $(\mathbb{Q} + i\mathbb{Q})$-linear combinations of $\{x_n\}_{n\in\mathbb{N}}$ is dense in $\mathcal{X}$. By density of $\mathbb{Q} + i\mathbb{Q}$ in $\mathbb{C}$

and Hahn-Banach theorem, it suffices to show: if $f \in \mathcal{X}^*$ vanishes on x_n for all n, then $f = 0$. To prove this, we notice that for all n,

$$\|f_n\| \leq 2|f(x_n) - f_n(x_n)| \leq 2\|f - f_n\|\|x_n\| = 2\|f - f_n\|,$$

thus

$$\|f\| \leq \|f - f_n\| + \|f_n\| \leq 3\|f - f_n\|.$$

Since $\{f_n\}$ is dense in $\mathcal{X}^*$, this shows $\|f\| = 0$, *i.e.* $f = 0$. $\square$