MATH 533, SPRING 2020, HW1

DUE IN CLASS, FEBRUARY 3

Problem 1. Let $\mathcal{E} = \{(a, b] : -\infty < a < b < \infty\}$. Prove the following.

(1) $\mathcal{E}$ is a base for a topology $\mathcal{T}$ on $\mathbb{R}$, in which the members of $\mathcal{E}$ are both open and closed.
(2) $\mathcal{T}$ is first countable but not second countable.
(3) $\mathbb{Q}$ is dense in $\mathbb{R}$ with respect to $\mathcal{T}$.

Problem 2. If X is a topological space, $x \in X$ is a *cluster point* of the sequence $\{x_j\}$ if for every neighborhood U of x, $x_j \in U$ for infinitely many j. Show that if X is first countable, x is a cluster point of $\{x_j\}$ iff some subsequence of $\{x_j\}$ converges to x.

Problem 3. Show that if X is a set, $\mathcal{F}$ a collection of real-valued functions on X, and $\mathcal{T}$ the weak topology generated by $\mathcal{F}$, then $\mathcal{T}$ is Hausdorff iff for every $x, y \in X$ with $x \neq y$ there exists $f \in \mathcal{F}$ with $f(x) \neq f(y)$.

Problem 4. Show that if A is a countable set and X_α is a first (resp. second) countable space for each $\alpha \in A$, then $\prod_{\alpha \in A} X_\alpha$ is first (resp. second) countable.

Problem 5. Prove that a Hausdorff topological space X is normal iff X satisfies the conclusion of Urysohn's lemma.

Problem 6. Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in a topological space, and for each $\alpha \in A$ let $E_\alpha = \{x_\beta : \beta \gtrsim \alpha\}$. Show that x is a cluster point of $\langle x_\alpha \rangle$ iff $x \in \bigcap_{\alpha \in A} \overline{E}_\alpha$.

Problem 7. Prove that if X has the weak topology generated by a family $\mathcal{F}$ of functions, then $\langle x_\alpha \rangle$ converges to $x \in X$ iff $\langle f(x_\alpha) \rangle$ converges to $f(x)$ for all $f \in \mathcal{F}$.

Problem 8. Let $(X, \mathcal{T})$ be a compact Hausdorff space. Prove that if $\mathcal{T}'$ is another topology on X which is strictly stronger than $\mathcal{T}$, then $\mathcal{T}'$ is Hausdorff but not compact. If $\mathcal{T}'$ is strictly weaker than $\mathcal{T}$, then $\mathcal{T}'$ is compact but not Hausdorff.

Problem 9. If $x \in [0,1]$, let $\sum_1^\infty a_n(x)2^{-n}$ $[a_n(x) = 0$ or $1]$ be the base-2 decimal expansion of x. Prove that the sequence $\langle a_n \rangle$ in $\{0,1\}^{[0,1]}$ has no pointwise convergent subsequence.

Problem 10. Prove the following.

(1) If (Y,ρ) is a metric space then
$$\rho^*(y,z) = \frac{\rho(y,z)}{1+\rho(y,z)}$$
is a bounded metric on Y which defines the same topology as ρ.

(2) If X is a topological space, the topology on $\mathbb{C}^X$ of uniform convergence is induced by the metric
$$\rho(f,g) = \sup_{x\in X} \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}.$$

(3) If X is a σ-compact LCH space and $\{U_n\}_1^\infty$ precompact open sets exhausting X, the topology on $\mathbb{C}^X$ of uniform convergence on compact sets is induced by the metric
$$\rho(f,g) = \sum_1^\infty \sup_{x\in \overline{U}_n} \frac{2^{-n}|f(x)-g(x)|}{1+|f(x)-g(x)|}.$$