MATH 322, SPRING 2019 MIDTERM 1, PRACTICE PROBLEMS

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Problem 1. Let $\|\cdot\|_2$ be the Euclidean norm on $\mathbb{R}^n$, $\|\underline{x}\|_2 = \sqrt{\sum_i x_i^2}$. Define the $2 \to 2$ operator norm on $\mathrm{Mat}_{n\times n}(\mathbb{R})$ by

$$\|M\|_{2\to 2} = \sup_{\|\underline{x}\|_2=1} \|M\underline{x}\|_2.$$

(1) Prove that the $2 \to 2$ operator norm is a norm on the vector space of $n \times n$ matrices.
(2) Given $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, prove that $\|AB\|_{2\to 2} \leq \|A\|_{2\to 2}\|B\|_{2\to 2}$.
(3) Let $p(x)$ be a power series of the real variable x, with radius of convergence r. Prove that if $\|A\|_{2\to 2} < r$ then the series defining $p(A)$ converges.
(4) Recall that a symmetric $n \times n$ matrix $A = A^t$ can be diagonalized $A = O^t DO$ where O is orthogonal, $O^t O = I_n$ and $D = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ is diagonal. Prove that $\|O\|_{2\to 2} = 1$, $\|D\|_{2\to 2} = \max |\lambda_i|$ and if $\max |\lambda_i| < r$ then
$$p(A) = O^t \,\mathrm{diag}(p(\lambda_1), ..., p(\lambda_n))O.$$

Problem 2. The *Cantor middle thirds set* is the set of all numbers in $[0, 1]$ which may be written as the sum

$$x = \sum_{n=0}^{\infty} \frac{x_n}{3^n}, \qquad x_n \in \{0, 2\}.$$

Prove that the Cantor middle thirds set has measure 0.

Problem 3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function, and let $E \subset \mathbb{R}^n$ be a set of measure 0. Prove that $f(E)$ has measure 0.

Problem 4. Let S_n denote the group of permutations on $\{1, 2, ..., n\}$. For $i < j$, the transposition $\tau_{i,j} \in S_n$ denotes the map $\tau_{i,j}(i) = j$, $\tau_{i,j}(j) = i$ and $\tau_{i,j}(k) = k$ otherwise.

(1) Prove that the set of transpositions $\{\tau_{i,j} : 1 \leq i < j \leq n\}$ generate the symmetric group.
(2) Given $\sigma \in S_n$, let $\iota(\sigma)$ denote the number of inversions in σ, that is, the number of $i < j$ such that $\sigma(i) > \sigma(j)$. Define the *sign* of a permutation, $\mathrm{sgn}(\sigma)$ to be $(-1)^{\iota(\sigma)}$. Prove that if τ_{i_1,j_1} and τ_{i_2,j_2} are two transpositions, then $\mathrm{sgn}(\tau_{i_1,j_1}) = -1$ and $\mathrm{sgn}(\tau_{i_1,j_1} \circ \tau_{i_2,j_2}) = 1$. (Hint: if $\sigma(k) = k$, then the number of $j < k$ with $\sigma(j) > k$ is equal to the number of $j > k$ with $\sigma(j) < k$.)
(3) Conclude that $\mathrm{sgn} : S_n \to \{-1, 1\}$ is a group homomorphism.
(4) Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be an $n \times n$ matrix. Prove that

$$\det A = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}$$

satisfies the axioms of the determinant.

Problem 5. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. Show that the power series

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$$

is defined for all t and is C^∞. Prove that

$$\frac{d}{dt} e^{tA} = A e^{tA}.$$

Explain how this can be used to solve a system of constant coefficient ODE's,

$$\begin{pmatrix} f' \\ f'' \\ \vdots \\ f^{(n+1)} \end{pmatrix} = A \begin{pmatrix} f \\ f' \\ \vdots \\ f^{(n)} \end{pmatrix}$$

with initial data

$$\begin{pmatrix} f(0) \\ f'(0) \\ \vdots \\ f^{(n)}(0) \end{pmatrix} = \underline{v}.$$

Problem 6. A function $f : X \to Y$ between two metric spaces is said to be Lipschitz if there is a constant $C > 0$ such that, for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2).$$

Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable with bounded derivative, then f is Lipschitz.