MATH 322, SPRING 2019 MIDTERM 2

APRIL 17

Solve problems 1 and 2, and two of problems 3-5. Each problem is worth 25 points.

Problem 1. Express in elementary alternating tensors

$$(x_1 + 2x_2 + 3x_3) \wedge (x_1 \wedge x_2 + x_2 \wedge x_3 - 3x_1 \wedge x_3).$$

Solution. We use the distributive property of the wedge product. Since an alternating tensor with repeated index is 0, the wedge product is

$$x_1 \wedge x_2 \wedge x_3 - 6x_2 \wedge x_1 \wedge x_3 + 3x_3 \wedge x_1 \wedge x_2 = 10x_1 \wedge x_2 \wedge x_3.$$

Problem 2. Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $\alpha(x,y) = \begin{pmatrix} x+y \\ x^2+y^2 \\ x^3+y^3 \end{pmatrix}$. Calculate $D\alpha$ and $V(D\alpha)$.

Solution. The derivative is $D\alpha(x,y) = \begin{pmatrix} 1 & 1 \\ 2x & 2y \\ 3x^2 & 3y^2 \end{pmatrix}$, and hence

$$D\alpha(x,y)^t D\alpha(x,y) = \begin{pmatrix} 1+4x^2+9x^4 & 1+4xy+9x^2y^2 \\ 1+4xy+9x^2y^2 & 1+4y^2+9y^4 \end{pmatrix}.$$

Hence,

$$V(D\alpha) = \sqrt{(1+4x^2+9x^4)(1+4y^2+9y^4) - (1+4xy+9x^2y^2)^2}.$$

Problem 3. Let $e_1, ..., e_n, f_1, ..., f_n$ be the standard basis vectors on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, and let $x_1, ..., x_n$, $y_1, ..., y_n$ be the dual basis. Given an alternating form g, define

$$g^{\wedge 1} = g, \qquad g^{\wedge(i+1)} = g \wedge g^{\wedge i}.$$

Let

$$\omega = x_1 \wedge y_1 + x_2 \wedge y_2 + \cdots + x_n \wedge y_n.$$

Calculate $\omega^{\wedge n}$. Partial credit will be awarded for calculating small n cases.

Solution. We'll show

$$\omega^{\wedge n} = n! x_1 \wedge y_1 \wedge \cdots \wedge x_n \wedge y_n = (-1)^{\frac{n(n-1)}{2}} n! x_1 \wedge \cdots \wedge x_n \wedge y_1 \wedge \cdots \wedge y_n.$$

The second equality holds since, to sort the altenating tensor into the given order, x_2 passes over one y index, x_3 passes over two, x_4 passes over three, etc, for a total of $1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$ sign changes.

To prove the first equality, write

$$\omega = \sum_{j=1}^{n} x_j \wedge y_j.$$

Hence, by linearity of the wedge product,

$$\omega^{\wedge n} = \sum_{j_1, j_2, \ldots, j_n = 1}^{n} x_{j_1} \wedge y_{j_1} \wedge x_{j_2} \wedge y_{j_2} \wedge \cdots \wedge x_{j_n} \wedge y_{j_n}.$$

The wedge product vanishes if there is a repeated index, so those non-vanishing terms have $j_1, ..., j_n$ a permutation of $1, 2, ..., n$. To sort such an alternating tensor into the order $x_1 \wedge y_1 \wedge \cdots \wedge x_n \wedge y_n$, an even number of swaps are made, since both x and y terms must be swapped. Hence each permutation obtains a positive term of $x_1 \wedge y_1 \wedge \cdots \wedge x_n \wedge y_n$ for a total of $n!$ terms.

Problem 4. Given a basis $e_1, e_2, ..., e_n$ for $\mathbb{R}^n$ with dual basis $x_1, x_2, ..., x_n$, define the elementary k symmetric tensors by

$$x_{i_1} \vee x_{i_2} \vee \cdots \vee x_{i_k} = \sum_{\sigma \in S_k} (x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k})^\sigma.$$

Prove that

$$\{x_{i_1} \vee \cdots \vee x_{i_k} : 1 \le i_1 \le i_2 \le \cdots \le i_k \le n\}$$

form a basis for symmetric k tensors on $\mathbb{R}^n$, and hence deduce that $\mathrm{sym}^k(\mathbb{R}^n)$ has dimension $\binom{n+k-1}{k}$. (Hint: how does the proof in the case of alternating k tensors change if the tensors are symmetric?)

Solution. Define an operator on k tensors, $S : \mathcal{L}^k(\mathbb{R}^n) \to \mathcal{L}^k(\mathbb{R}^n)$ by

$$Sf = \frac{1}{k!} \sum_{\sigma \in S_k} f^\sigma.$$

If f is symmetric, then $Sf = f$. Also, for any k tensor f, and any $\tau \in S_k$,

$$\begin{aligned}(Sf)^\tau &= \left(\frac{1}{k!} \sum_{\sigma \in S_k} f^\sigma\right)^\tau \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} (f^\sigma)^\tau \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} f^{\tau \circ \sigma} = Sf.\end{aligned}$$

Hence, Sf is symmetric. Similarly,

$$\begin{aligned}S(f^\tau) &= \frac{1}{k!} \sum_{\sigma \in S_k} (f^\tau)^\sigma \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} f^{\sigma \circ \tau} = S(f).\end{aligned}$$

It follows that if $f = x_{i_1} \otimes \cdots \otimes x_{i_k}$ and if $g = x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(k)}}$ for some permutation $\sigma \in S_k$, then $Sf = Sg$.

Let f be a symmetric tensor in S_k, and write

$$f = \sum_I b_I x_{i_1} \otimes \cdots \otimes x_{i_k}.$$

Then

$$f = Sf = \sum_I b_I S(x_{i_1} \otimes \cdots \otimes x_{i_k}).$$

This shows that the elementary symmetric tensors span.

To check that the elementary symmetric tensors are linearly independent, calculate

$$\begin{aligned} x_{i_1} \vee \cdots \vee x_{i_k}(e_{j_1}, ..., e_{j_k}) &= \sum_{\sigma \in S_k} x_{i_1} \otimes \cdots \otimes x_{i_k}(e_{j_{\sigma(1)}}, ..., e_{j_{\sigma(k)}}) \\ &= \begin{cases} n_1! n_2! \cdots n_r! & I = J \\ 0 & I \neq J \end{cases} \end{aligned}$$

where $n_1, ..., n_r$ are the multiplicities of the indices in I. Hence if

$$f = \sum_I b_I x_{i_1} \vee \cdots \vee x_{i_k} = 0$$

then

$$0 = f(e_{j_1}, ..., e_{j_k}) = n_1! \cdots n_r! b_J = 0$$

so $b_J = 0$ for all J.

Problem 5. For $p > 0$ define the ℓ^p norm on $\mathbb{R}^n$ by $\|\underline{x}\|_p^p = x_1^p + \cdots + x_n^p$. Define $B_{n,p}(t) = \{\underline{x} \in \mathbb{R}^n_{>0} : \|\underline{x}\|_p < t\}$. Let $\alpha : B_{n-1,p}(1) \times (0,t)$ be the coordinate patch

$$\alpha(\underline{x}, s) = \begin{pmatrix} sx_1 \\ sx_2 \\ \vdots \\ sx_{n-1} \\ s(1 - x_1^p - \cdots - x_{n-1}^p)^{\frac{1}{p}} \end{pmatrix}.$$

a. Calculate

$$V(D\alpha) = \frac{s^{n-1}}{(1 - x_1^p - \cdots - x_{n-1}^p)^{\frac{p-1}{p}}},$$

and, hence,

$$V(B_{n,p}(1)) = \frac{1}{n} \int_{\underline{x} \in B_{n-1,p}(1)} \frac{1}{(1 - x_1^p - \cdots - x_{n-1}^p)^{\frac{p-1}{p}}}.$$

b. Determine $V(B_{n,p}(1))$ by calculating $\left(\int_{\mathbb{R}_{>0}} e^{-x^p} dx\right)^n$ in two ways. Express your answer in terms of the Gamma function, which is defined in $\Re(s) > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx.$$

Solution.

a. We calculate, for $d = (1 - x_1^p - \cdots - x_{n-1}^p)^{\frac{1}{p}}$,

$$D\alpha(\underline{x}, s) = \begin{pmatrix} s & 0 & \cdots & 0 & x_1 \\ 0 & s & \cdots & 0 & x_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & s & x_{n-1} \\ -s\frac{x_1^{p-1}}{d^{p-1}} & -s\frac{x_2^{p-1}}{d^{p-1}} & \cdots & -s\frac{x_{n-1}^{p-1}}{d^{p-1}} & d \end{pmatrix}.$$

Since $D\alpha$ is $n \times n$, $V(D\alpha) = |\det D\alpha|$. In the determinant, pull out a factor of s from each of the first $n-1$ columns, and a factor of $d^{-(p-1)}$

from the last row, to obtain

$$V(D\alpha(\underline{x}, s)) = \frac{s^{n-1}}{(1 - x_1^p - \cdots - x_{n-1}^p)^{\frac{p-1}{p}}}$$
$$\times \left| \det \begin{pmatrix} 1 & 0 & \cdots & 0 & x_1 \\ 0 & 1 & \cdots & 0 & x_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & x_{n-1} \\ -x_1^{p-1} & -x_2^{p-1} & \cdots & -x_{n-1}^{p-1} & 1 - x_1^p - \cdots - x_{n-1}^p \end{pmatrix} \right|.$$

The latter determinant is 1, since row operations can be performed to clear the last column to the standard basis vector e_n. Using α with $B_{n-1,p}(1) \times (0,1)$ as a coordinate patch for $B_{n,p}(1)$ gives the volume of $B_{n,p}(1)$ as

$$\int_{B_{n-1,p}(1)\times(0,1)} V(d\alpha) = \int_{B_{n-1,p}(1)\times(0,1)} \frac{s^{n-1}}{(1 - x_1^p - \cdots - x_{n-1}^p)^{\frac{p-1}{p}}}.$$

Applying Fubini's theorem and integrating s from 0 to 1 gives the claimed formula.

b. First substitute $u = x^p$, $du = px^{p-1}dx$ to calculate

$$\int_0^\infty e^{-x^p} dx = \frac{1}{p} \int_0^\infty u^{\frac{1}{p}} e^{-u} \frac{du}{u} = \frac{1}{p} \Gamma\left(\frac{1}{p}\right).$$

It follows that

$$\begin{aligned} \left(\frac{1}{p}\Gamma\left(\frac{1}{p}\right)\right)^n &= \left(\int_0^\infty e^{-x^p} dx\right)^n \\ &= \int_{\mathbb{R}^n_{>0}} e^{-x_1^p - \cdots - x_n^p} d\underline{x} \\ &= \int_{B_{n-1,p}(1)\times(0,\infty)} \frac{s^{n-1}e^{-s^p}}{(1 - x_1^p - \cdots - x_{n-1}^p)^{\frac{p-1}{p}}} \\ &= nV(B_{n,p}(1)) \int_0^\infty s^n e^{-s^p} \frac{ds}{s}. \end{aligned}$$

After the same change of variables as before, the last integral is

$$\frac{1}{p}\int_0^\infty u^{\frac{n}{p}}e^{-u}\frac{du}{u} = \frac{1}{p}\Gamma\left(\frac{n}{p}\right).$$

It follows that

$$\left(\frac{1}{p}\Gamma\left(\frac{1}{p}\right)\right)^n = V(B_{n,p}(1))\frac{n}{p}\Gamma\left(\frac{n}{p}\right).$$

or

$$V(B_{n,p}(1)) = \frac{\left(\frac{1}{p}\Gamma\left(\frac{1}{p}\right)\right)^n}{\frac{n}{p}\Gamma\left(\frac{n}{p}\right)} = \frac{\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)}.$$