MATH 322, SPRING 2019 MIDTERM 1

MARCH 4

Each problem is worth 20 points.

Problem 1. Prove that any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^n$ are equivalent, in the sense that there are constants $C_1, C_2 > 0$ such that, for any $\underline{v} \in V$,

$$C_1\|\underline{v}\|_1 \leq \|\underline{v}\|_2 \leq C_2\|\underline{v}\|_1.$$

(Hint: it suffices to assume that $\|\cdot\|_2$ is the usual Euclidean norm. Check that $\|\cdot\|_1$ is continuous with respect to $\|\cdot\|_2$ by considering $\|\cdot\|_1$ on the standard basis. Then consider $\|\cdot\|_1$ on the unit sphere in the $\|\cdot\|_2$ norm.)

Solution 1. By the triangle inequality, and then Cauchy-Schwarz,

$$\begin{aligned}\|a_1\underline{e}_1 + \cdots + a_n\underline{e}_n\|_1 &\leq \|a_1\underline{e}_1\|_1 + \cdots + \|a_n\underline{e}_n\|_1 \\ &\leq |a_1|\|\underline{e}_1\|_1 + \cdots + |a_n|\|\underline{e}_n\|_1 \\ &\leq \sqrt{a_1^2 + \cdots + a_n^2}\sqrt{\|\underline{e}_1\|_1^2 + \cdots + \|\underline{e}_n\|_1^2} \\ &\leq C\|\underline{a}\|_2\end{aligned}$$

for $C = \sqrt{\|\underline{e}_1\|_1^2 + \cdots + \|\underline{e}_n\|_1^2}$. By the triangle inequality again,

$$|\|\underline{x}\|_1 - \|\underline{y}\|_1| \leq \|\underline{x} - \underline{y}\|_1 \leq C\|\underline{x} - \underline{y}\|_2.$$

This proves that $\|\cdot\|_1$ is continuous as a function on $\mathbb{R}^n$ with the topology generated by $\|\cdot\|_2$. Since the sphere

$$S = \{\underline{x} \in \mathbb{R}^n : \|\underline{x}\|_2 = 1\}$$

is closed and bounded and hence compact in this topology, and $\|\cdot\|_1$ is continuous, $\|\cdot\|_1$ achieves its min m and max M on S. Since for $\underline{x} \in S$, $\underline{x} \neq \underline{0}$, $M \geq m > 0$, and hence, for any $\underline{0} \neq \underline{v} \in \mathbb{R}^n$, $\underline{w} = \frac{v}{\|\underline{v}\|_2}$,

$$m \leq \|\underline{w}\|_1 = \frac{\|\underline{v}\|_1}{\|\underline{v}\|_2} \leq M.$$

Problem 2. Prove the following statement from lecture. Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}^m$ be C^2 on A. Then for all $1 \leq i, j \leq n$, $D_i D_j f = D_j D_i f$.

Solution 2. See Munkres, pp.52-54.

Problem 3. The $n \times n$ orthogonal group O_n is defined implicitly by

$$O_n = \{A \in \mathrm{Mat}_{n\times n} : A^t A - I_n = 0\}.$$

Prove that there is a neighborhood U of $I_n \in O_n$, a neighborhood V of

$$\underline{0} \in \mathbb{R}^{\frac{n(n-1)}{2}} = \{\epsilon_{i,j}, 1 \le i < j \le n : \epsilon_{ij} \in \mathbb{R}\}$$

and a C^∞ surjective map $f : V \to U$ given by

$$f(\underline{\epsilon}) = \begin{pmatrix} f_{1,1}(\underline{\epsilon}) & \epsilon_{1,2} & \epsilon_{1,3} & \cdots & \epsilon_{1,n} \\ f_{2,1}(\underline{\epsilon}) & f_{2,2}(\underline{\epsilon}) & \epsilon_{2,3} & \cdots & \epsilon_{2,n} \\ f_{3,1}(\underline{\epsilon}) & f_{3,2}(\underline{\epsilon}) & f_{3,3}(\underline{\epsilon}) & & \epsilon_{3,n} \\ \vdots & \vdots & & \ddots & \vdots \\ f_{n,1}(\underline{\epsilon}) & f_{n,2}(\underline{\epsilon}) & f_{n,3}(\underline{\epsilon}) & \cdots & f_{n,n}(\underline{\epsilon}) \end{pmatrix}.$$

Solution 3. Let $A = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,n} \end{pmatrix} \in \mathbb{R}^{n\times n}$. Hence $M = A^t A$ has entries

$$M_{i,j} = \sum_{k=1}^n x_{k,i} x_{k,j}.$$

It follows that

$$\frac{\partial M_{i,j}}{\partial x_{k,l}} = \delta_{l=i} x_{k,j} + \delta_{l=j} x_{k,i}.$$

At the identity $A = I_n$, $x_{i,i} = 1$ and $x_{i,j} = 0$ if $i \ne j$, and hence, at the identity,

$$\frac{\partial M_{i,j}}{\partial x_{k,l}}(I_n) = \delta_{l=i}\delta_{k=j} + \delta_{l=j}\delta_{k=i}.$$

Since $F(A) = A^t A - I_n$ is symmetric, treat this as a map into the lower triangular coordinates, $i \ge j$. At the identity, $DF(I_n)$ has column corresponding to $x_{i,j}$ with $i \ge j$ equal to the standard basis vector at position i, j if $i > j$, and twice this vector if $i = j$. In particular, the minor of $DF(I_n)$ corresponding to the coordinates $x_{i,j}$ with $i \ge j$ is invertible. Since the map F is a polynomial, hence C^∞, the implicit function theorem applies to give an open set $V \subset \mathbb{R}^{\frac{n(n-1)}{2}}$ containing 0 identified with the coordinates $\{\epsilon_{i,j}\}$,

$1 \leq i < j \leq n$, and an open set $U \subset \mathrm{Mat}_{n\times n}(\mathbb{R})$ containing the identity, and a C^∞ map $f : V \to \mathbb{R}^{\frac{n(n+1)}{2}}$ identified with $\{x_{i,j}\}$, $1 \leq j \leq i \leq n$ such that f maps V onto $O_n \cap U$ as described in the problem statement.

Problem 4. Let $A \subset \mathbb{R}^n$ be open, containing 0, and let $f : A \to \mathbb{R}$ be C^r. Prove the following multivariable Taylor's formula with integral remainder. If $\underline{x} \in \mathbb{R}^n$ is such that the line segment between $\underline{0}$ and $\underline{x}$ is contained in A, then

$$f(\underline{x}) = \sum_{k=0}^{r-1} \sum_{j_1+\cdots+j_n=k} \frac{D_1^{j_1} \cdots D_n^{j_n} f(\underline{0}) x_1^{j_1} \cdots x_n^{j_n}}{j_1! \cdots j_n!} + \int_0^1 \frac{d^r}{dt^r} f(t\underline{x}) \frac{(1-t)^{r-1}}{(r-1)!} dt.$$

(Hint: first calculate $\frac{d^k}{dt^k} f(t\underline{x})$. You may assume without proof the one dimensional version of Taylor's formula with integral remainder.)

Solution 4. The function $g(t) = f(t\underline{x})$ is C^r by the chain rule. Treating this function as a function of the single variable t,

$$g(1) = f(\underline{x}) = \sum_{k=0}^{r-1} \frac{g^{(k)}(\underline{0})}{k!} t^k + \int_0^1 \frac{g^{(r)}(t)}{(r-1)!} (1-t)^{r-1} dt$$

follows from the 1-dimensional Taylor's theorem with integral remainder, so it suffices to prove that

$$\frac{g^{(k)}(t)}{k!} = \sum_{j_1+\cdots+j_n=k} \frac{D_1^{j_1} \cdots D_n^{j_n} f(t\underline{x}) x_1^{j_1} \cdots x_n^{j_n}}{j_1! \cdots j_n!}.$$

This is proved by induction.

The base case $k = 0$ is trivial.

Assuming the validity for some $0 \leq k < r$, note that

$$\frac{k!}{j_1! \cdots j_n!}$$

is the number of words of length k containing letters of frequency $j_1, j_2, ..., j_n$. Since the mixed partials of order at most r commute, the identity may thus be rewritten

$$g^{(k)}(t) = \sum_{s_1, s_2, \ldots, s_k \in \{1,2,\ldots,n\}} D_{s_k} D_{s_{k-1}} \cdots D_{s_1} f(t\underline{x}) x_{s_1} \cdots x_{s_k}.$$

For differentiable H, the chain rule implies

$$\frac{d}{dt}H(t\underline{x}) = DH(t\underline{x})\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = D_1H(t\underline{x})x_1 + \cdots + D_nH(t\underline{x})x_n.$$

Applying this to each term in the sum of $g^{(k)}(t)$,

$$g^{(k+1)}(t) = \sum_{s_{k+1}=1}^{n} \sum_{s_1,\ldots,s_k\in\{1,\ldots,n\}} D_{s_{k+1}}D_{s_k}\cdots D_{s_1}f(t\underline{x})x_{s_1}\cdots x_{s_k}x_{s_{k+1}},$$

which proves the inductive step.