Each problem is worth 10 points.
Problem 1. Let the power set of a set $S$ be $P(S) = \{ A : A \subset S \}$, the set of all subsets of $S$. Prove by induction that if $S$ has $n \geq 1$ elements, then $P(S)$ has $2^n$ elements.

Solution. Base case ($n = 1$): If $|S| = 1$ the power set is $\{ \emptyset, S \}$.

Inductive step: Let $n \geq 1$ and assume the claim holds for all sets of size $n$. Let $S$ be a set of size $n + 1$ and let $x \in S$. Split the power set of $S$ into subsets that contain $x$ and subsets that do not. Those that do not form the power set of $S \setminus \{x\}$, and hence there are $2^n$ of these. Those that contain $x$ can be found by appending $x$ to each subset in the power set of $S \setminus \{x\}$, for another $2^n$ subsets. Thus there are $2^{n+1}$ elements in the power set of $S$. 
Problem 2. Let $K := \{ s + t\sqrt{2} : s, t \in \mathbb{Q} \}$. Show that $K$ satisfies the following

a. If $x_1, x_2 \in K$, then $x_1 + x_2 \in K$ and $x_1 x_2 \in K$.

b. If $x \neq 0$ and $x \in K$, then $\frac{1}{x} \in K$.

Solution. a. Let $x_1 = (s_1 + t_1\sqrt{2})$ and $x_2 = (s_2 + t_2\sqrt{2})$, $s_1, s_2, t_1, t_2 \in \mathbb{Q}$. Then $x_1 + x_2 = (s_1 + s_2) + (t_1 + t_2)\sqrt{2}$ shows $x_1 + x_2 \in K$. Also $x_1 x_2 = (s_1 + t_1\sqrt{2})(s_2 + t_2\sqrt{2}) = (s_1s_2 + 2t_1t_2) + (s_1t_2 + s_2t_1)\sqrt{2}$ shows $x_1 x_2 \in K$.

b. Let $x = s + t\sqrt{2}$ with not both $s, t = 0$. Then $\frac{1}{x} = \frac{s-t\sqrt{2}}{s^2-2t^2}$. Notice the denominator does not vanish since 2 is not a square. Since $\frac{s}{s^2-2t^2}$ and $\frac{t}{s^2-2t^2}$ are both rational, this completes the proof.
Problem 3. Calculate the following limits.

a. Using only the definition of limits, calculate the limit \( \lim_{n \to \infty} \frac{4n^2 + 3}{2n^2 + 1} \).

b. If \( 0 < a < b \), determine \( \lim \left( \frac{a^{n+1} + b^{n+1}}{a^n + b^n} \right) \). (You do not have to use the definition and you may rely on properties of the limit.)

Solution. a. Write \( \frac{4n^2 + 3}{2n^2 + 1} = 2 + \frac{1}{2n^2 + 1} \). We show the limit is 2. We need to show that for \( \epsilon > 0 \) we can find \( N \) so that \( n > N \) implies \( \frac{1}{2n^2 + 1} < \epsilon \). It suffices to choose \( N = \frac{1}{\sqrt{\epsilon}} \), which implies

\[
\left| \frac{4n^2 + 3}{2n^2 + 1} - 2 \right| = \frac{1}{2n^2 + 1} < \frac{1}{\epsilon + 1} < \epsilon.
\]

b. Write \( \frac{a^{n+1} + b^{n+1}}{a^n + b^n} = b \frac{1 + \left( \frac{a}{b} \right)^{n+1}}{1 + \left( \frac{a}{b} \right)^n} \). Thus

\[
\lim \frac{a^{n+1} + b^{n+1}}{a^n + b^n} = b \frac{1 + \lim \left( \frac{a}{b} \right)^{n+1}}{1 + \lim \left( \frac{a}{b} \right)^n} = b.
\]
**Problem 4.** Establish the convergence or divergence of the sequence \((y_n)\) where

\[ y_n := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}, \quad n \in \mathbb{N}. \]

**Solution.** We have

\[ y_{n+1} - y_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0. \]

Thus the sequence \(y_n\) is increasing. It is also bounded above by 1 since each of the \(n\) terms in the sum defining \(y_n\) has size at most \(\frac{1}{n+1}\). Thus it converges.
Problem 5. Show that if $x_n > 0$ for all $n \in \mathbb{N}$, then $\lim x_n = 0$ if and only if $\lim \frac{1}{x_n} = \infty$.

Solution. Suppose $\lim x_n = 0$. Given $M > 0$ choose $N$ so that $n > N$ implies $x_n < \frac{1}{M}$ so $\frac{1}{x_n} > M$ and this proves $\lim \frac{1}{x_n} = \infty$. Conversely if $\lim \frac{1}{x_n} = \infty$, given $\epsilon > 0$ choose $N$ so that $n > N$ implies $\frac{1}{x_n} > \frac{1}{\epsilon}$. Then $0 < x_n < \epsilon$ and so $\lim x_n = 0$. 