# MATH 320, FALL 2017 PRACTICE MIDTERM 2

NOVEMBER 7

Each problem is worth 10 points.

**Problem 1.**

a. (4 points) State the Intermediate Value Theorem.

b. (6 points) Prove that a continuous function $f : [0,1] \to [0,1]$ has a fixed point $x$, satisfying $f(x) = x$.

**Solution.**

a. Let $f$ be continuous on the interval $[a, b]$. For all $y$ between $f(a)$ and $f(b)$ there is a point $c \in [a, b]$ such that $f(c) = y$.

b. Let $g(x) = f(x) - x$, which is continuous on $[0, 1]$. Solving $f(x) = x$ is equivalent to solving $g(x) = 0$. If $g(0) = 0$ or $g(1) = 0$ then we are done, so assume otherwise. Then $g(0) > 0$ and $g(1) < 0$, so that the Intermediate Value Theorem implies that there is $c \in (0, 1)$ such that $g(c) = 0$.

**Problem 2.**

a. (3 points) Let $f$ be a real valued function on a metric space $(S, d)$. State one of the two equivalent definitions of continuity of $f$ at a point $x \in S$.

b. (7 points) Prove that a continuous function on a closed bounded interval $[a, b]$ is bounded.

**Solution.**

a. $f$ is continuous at $x$ if, for all $\epsilon > 0$ there is $\delta > 0$ such that if $y \in S$ and $d(x, y) < \delta$, then $|f(x) - f(y)| < \epsilon$.

b. Suppose that $f$ is unbounded. Then for each integer $n \geq 1$ there exists a point $x_n \in [a, b]$ such that $|f(x_n)| > n$. By the Bolzano-Weierstrass Theorem, there is a convergent subsequence $\{x_{n_k}\}$, converging to $x \in [a, b]$. Then $f(x_{n_k}) \to f(x)$ as $k \to \infty$ by continuity, so $|f(x_{n_k})| \to |f(x)|$, but this violates the fact that $f(x_{n_k})$ is unbounded.

**Problem 3.** (10 points) State the alternating series test. Using this, or otherwise, prove that the limit

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n}-\log N\right)$$

exists and is finite. (Remark: this number is called Euler's constant.)

**Solution.** The alternating series test states that if $\{a_n\}$ is a decreasing sequence of non-negative real numbers such that $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges to a limit $a$, and the partial sums

$$S_N=\sum_{n=1}^{N}(-1)^{n+1}a_n$$

satisfy $|S_N - a| \le a_{N+1}$.

To evaluate the limit, write $\log N = \int_1^N \frac{dt}{t}$. We give two methods to prove that the limit exists. First, write

$$\sum_{n=1}^{N}\frac{1}{n}-\log N=\sum_{n=1}^{N-1}\left(\frac{1}{n}-\int_n^{n+1}\frac{dt}{t}\right)+\frac{1}{N}.$$

Since $\lim \frac{1}{N}=0$, it suffices to prove that $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\int_n^{n+1}\frac{dt}{t}\right)$ converges. Note that $a_n=\frac{1}{n}-\int_n^{n+1}\frac{dt}{t}$ is a sequence of positive terms, so it suffices to prove that its sequence of partial sums is bounded. Since $0\le a_n\le \frac{1}{n}-\frac{1}{n+1}$,

$$\sum_{n=1}^{N-1}a_n\le\sum_{n=1}^{N-1}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{N},$$

which is bounded.

To use the alternating series test instead, define a sequence

$$a_{2n-1}=\int_{n-\frac{1}{2}}^{n}\frac{1}{t}-\frac{1}{n}dt,\qquad a_{2n}=\int_n^{n+\frac{1}{2}}\frac{1}{n}-\frac{1}{t}dt.$$

By splitting the integral into pieces of length $\frac{1}{2}$,

$$\sum_{n=1}^{N}\frac{1}{n}-\int_1^N\frac{dt}{t}=1-\int_1^{\frac{3}{2}}\frac{dt}{t}+\sum_{n=2}^{N-1}(-a_{2n-1}+a_{2n})+\frac{1}{N}-\int_{N-\frac{1}{2}}^{N}\frac{dt}{t}.$$

Notice that $\int_{N-\frac{1}{2}}^{N}\frac{dt}{t}<\frac{1}{2N-1}$, which tends to 0 as $N\to\infty$, so it suffices to prove that the alternating series $\sum_{n=2}^{\infty}(-a_{2n-1}+a_{2n})$ converges. Since $a_n\geq 0$ and $a_n\to 0$, it suffices to check that $a_n$ is decreasing.

We use several times the that, for $0<\delta<x$,

$$\frac{1}{x-\delta}+\frac{1}{x+\delta}=\frac{2x}{x^2-\delta^2} \tag{1}$$

is increasing in $\delta$. Write

$$a_{2n-1}-a_{2n}=\int_0^{\frac{1}{2}}\frac{1}{n-t}-\frac{2}{n}+\frac{1}{n+t}dt=\int_0^{\frac{1}{2}}\frac{2n}{n^2-t^2}-\frac{2}{n}dt>0.$$

Also

$$\begin{aligned}a_{2n}-a_{2n+1}&=\int_0^{\frac{1}{2}}\left(\frac{1}{n}-\frac{1}{n+\frac{1}{2}-t}\right)-\left(\frac{1}{n+\frac{1}{2}+t}-\frac{1}{n+1}\right)dt\\&=\int_0^{\frac{1}{2}}\frac{1}{n}+\frac{1}{n+1}-\frac{1}{n+\frac{1}{2}-t}-\frac{1}{n+\frac{1}{2}+t}dt.\end{aligned}$$

The integrand is non-negative by considering $x=n+\frac{1}{2}$ and comparing $\delta=\frac{1}{2}$ and $\delta=t$, $0\leq t\leq\frac{1}{2}$ in (1).

**Problem 4.**

a. (3 points) State the definition of a countable set $S$.
b. (7 points) Prove that the set of sequences $\{a_n\}_{n\in\mathbb{N}}$ with values in $\{0,1\}$ is uncountable.

**Solution.**

a. The set $S$ is countable if there is an onto map $f : \mathbb{N} \to S$. Equivalently, $S$ is countable if there is an one-to-one map $g : S \to \mathbb{N}$.
b. Suppose for contradiction that there is an onto map $f$ from $\mathbb{N}$ to the set $S$ of sequences taking values in $\{0,1\}$. Indicate the image of $k \in \mathbb{N}$ under this map by $\left\{a_n^{(k)}\right\}_{n\in\mathbb{N}}$. Define sequence $\{a_n\}_{n\in\mathbb{N}}$ by $a_n = 1 - a_n^{(n)}$, which takes values in $\{0,1\}$. Then for all $k$, $\{a_n\}_{n\in\mathbb{N}} \neq \left\{a_n^{(k)}\right\}_{n\in\mathbb{N}}$ since $a_k \neq a_k^{(k)}$. Hence $\{a_n\}_{n\in\mathbb{N}}$ is not in the image of $f$, a contradiction.