MATH 320, FALL 2017 PRACTICE FINAL EXAM

DECEMBER 15

Each problem is worth 10 points.

Problem 1.

a. (4 points) State the definition of a sequence of functions on an interval $[a, b]$ which converges uniformly to a function f.

b. (6 points) Prove that a sequence of functions $\{f_n\}$ on an interval $[a, b]$ which is uniformly Cauchy converges uniformly to a limit function f on $[a, b]$.

Problem 2. A function f on $[a, b]$ is said to be convex on $[a, b]$ if for any $a \leq x < y \leq b$ and for any $0 \leq t \leq 1$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

a. (4 points) Prove that if f is convex on $[a, b]$, then for any $a \leq x < y \leq z < w \leq b$,

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(w) - f(z)}{w - z}.$$

b. (6 points) Prove that if f is convex on $[a, b]$, then it is integrable there. (Hint: you may use, without proof, that an increasing function on an interval $[a, b]$ is integrable.)

Problem 3.

a. (7 points) A sequence $\{a_n\}_{n=0}^{\infty}$ is defined recursively by

$$a_0 = 0, \qquad a_1 = 1$$
$$a_{n+1} = 5a_n - 6a_{n-1}, \qquad n \geq 1.$$

Define $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Find a closed form expression for $f(x)$ and determine its radius of convergence.

b. (3 points) Determine, with proof, the value a_{1000}.

Problem 4.

a. (4 points) Let f be defined on $[0,1]$ by $f(x) = 1$ if x is rational, $f(x) = 0$ otherwise. Prove that f is not Riemann integrable.

b. (6 points) Let f be defined on $[0,1]$ by $f(x) = \frac{1}{q}$ if $x = \frac{p}{q}$ is rational in lowest terms, $f(x) = 0$ otherwise. Prove that $\int_0^1 f(x)dx = 0$.

Problem 5. Determine the following limits.

a. (5 points)

$$\lim_{x\to 0} \frac{\cos x - 1}{x - \log(1+x)}.$$

b. (5 points)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \frac{1}{1 + \left(\frac{j}{N}\right)^2}.$$

Problem 6.

a. (4 points) Find the degree 3 Taylor polynomial of e^{e^x-1} about $x = 0$.

b. (6 points) Prove that the radius of convergence of the Taylor series for e^{e^x-1} is at least 1.

Hint: Define a sequence of polynomials $P_n(u)$ such that $\left(\frac{d}{dx}\right)^n e^{e^x} = e^{e^x} P_n(e^x)$. $P_n(1)$ is the sum of the coefficients.

(The actual radius of convergence is ∞. This is most easily checked with complex analysis.)