MATH 320, FALL 2017 MIDTERM 2

NOVEMBER 7

Each problem is worth 10 points.

Problem 1.

a. (3 points) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Define

$$\limsup_{n\to\infty} a_n.$$

b. (7 points) Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be sequences of real numbers. Assume that $\limsup a_n$ and $\limsup b_n$ are finite. Prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Give an example where equality does not hold.

Problem 2.

a. (4 points) State the definition of a metric d on a set S.

b. (6 points) Given two points $\underline{x} = (x_1, ..., x_n)$ and $\underline{y} = (y_1, ..., y_n)$ in $\mathbb{R}^n$, the ℓ^1 and ℓ^∞ distances between $\underline{x}$ and $\underline{y}$ are

$$d_1(\underline{x}, \underline{y}) = \sum_{i=1}^{n} |x_i - y_i|, \qquad d_\infty(\underline{x}, \underline{y}) = \max\{|x_i - y_i|, i = 1, ..., n\}.$$

Check that the ℓ^1 and ℓ^∞ distances are metrics on $\mathbb{R}^n$, then check that a sequence $\{\underline{x}_k\}_{k\in\mathbb{N}}$ of elements of $\mathbb{R}^n$ converges in the ℓ^1 metric if and only if it converges in the ℓ^∞ metric.

Problem 3. The binomial coefficients are defined for integers $0 \leq k \leq n$ by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

a. (5 points) Decide, with proof, whether the series $\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}}$ converges.

b. (5 points) Prove that $\frac{\binom{2n}{n}}{2^{2n}} \to 0$.

$\Big[$Hint: first check that $\frac{\binom{2n}{n}}{2^{2n}} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \dots \cdot \frac{1}{2}$.$\Big]$

Problem 4. (10 points) Prove that a continuous function on a closed bounded interval $[a, b]$ is uniformly continuous.