MATH 311, FALL 2020 FINAL

DECEMBER 9

Each problem is worth 10 points.

Problem 1. Let p be prime, $n > 1$ and $q = p^n$. Recall that if $P(x)$ is a irreducible degree n polynomial over the finite field $\mathbb{F}_p$ then $\mathbb{F}_p[x]/(P(x))$ is a field with $q = p^n$ elements.

a. Let $\mathbb{F}_q$ be any field with $q = p^n$ elements. Prove that $\mathbb{F}_q$ has characteristic p, that is, $p \cdot x = 0$ for all x, and conclude that $\mathbb{F}_q$ has $\mathbb{F}_p$ as a subfield.
b. Prove that $x^q - x$ factors into distinct linear factors over $\mathbb{F}_q$, equivalently, that $x^q - x = 0$ is true for all $x \in \mathbb{F}_q$. Using this or otherwise, prove that all fields of order $\mathbb{F}_q$ are isomorphic.
c. Prove that $\mathbb{F}_q^\times$, the non-zero elements of $\mathbb{F}_q$, form a cyclic group.
d. Let $f(d)$ be the number of degree d monic irreducible polynomials over $\mathbb{F}_p$. Prove $p^n = \sum_{d|n} df(d)$. (Hint: associate to each element of $\mathbb{F}_q$ its minimal polynomial over $\mathbb{F}_p$.)
e. Prove
$$f(n) = \frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right) p^d = \frac{p^n}{n} + O\left(p^{\frac{n}{2}}\right).$$
(Remark: if $x = p^n$, an estimate of this quality for the number of prime numbers less than x is equivalent to the Riemann Hypothesis.)

In the next two problems you may assume the *Poisson summation formula* in the following form. Let $A < B$ be integers, and let f be differentiable on (A, B) and continuous on $[A, B]$. Then

$$\frac{1}{2}f(A)+f(A+1)+f(A+2)+\cdots+f(B-1)+\frac{1}{2}f(B) = \sum_{\nu=-\infty}^{\infty} \int_A^B f(x)e^{2\pi i\nu x}dx.$$

Problem 2. Let $N \geq 1$ be an integer, and let S be the Gauss sum $\sum_{n=0}^{N-1} e^{\frac{2\pi i n^2}{N}}$. Prove the evaluation

$$S = \begin{cases} (1+i)N^{\frac{1}{2}} & N \equiv 0 \bmod 4 \\ N^{\frac{1}{2}} & N \equiv 1 \bmod 4 \\ 0 & N \equiv 2 \bmod 4 \\ iN^{\frac{1}{2}} & N \equiv 3 \bmod 4 \end{cases}$$

by filling in the following steps:

a. Use the Poisson summation formula to write

$$S = \sum_{\nu=-\infty}^{\infty} \int_0^N e^{2\pi i\nu x + 2\pi i \frac{x^2}{N}} dx.$$

b. Complete the square in the exponent to write

$$S = N \sum_{\nu=-\infty}^{\infty} e^{-\frac{1}{2}\pi i N\nu^2} \int_{\frac{\nu}{2}}^{1+\frac{\nu}{2}} e^{2\pi i N y^2} dy.$$

c. Separate the odd and even terms, writing each as an infinite integral. It may help to use the known value when $N = 1$.

Problem 3. The Γ function is defined for $s \in \mathbb{C}$, $\Re(s) > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-x} x^s \frac{dx}{x}.$$

Let $\omega(x) = \sum_{n=1}^\infty e^{-n^2\pi x}$ and $\theta(x) = 2\omega(x) + 1 = \sum_{n\in\mathbb{Z}} e^{-n^2\pi x}$.

a. Prove, for $x > 0$, $\theta(x^{-1}) = x^{\frac{1}{2}}\theta(x)$.

b. Prove, for $\Re(s) > 1$,

$$\xi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\omega(x)dx.$$

c. Prove

$$\xi(s) = \frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}-1}\right)\omega(x)dx.$$

Prove that the latter integral is analytic for all $s \in \mathbb{C}$. This gives the meromorphic continuation of $\zeta(s)$ to $\mathbb{C}$. Prove $\xi(s) = \xi(1-s)$.

Problem 4. The Bernoulli numbers are defined by

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!}.$$

a. Calculate $B_1, B_2, B_3, B_4, B_5, B_6$.

b. Take the logarithmic derivative of the infinite product formula

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$

to obtain

$$z \cot z = 1 + 2\sum_{n=1}^{\infty} \frac{z^2}{z^2 - n^2\pi^2} = 1 - 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{z^{2k}}{n^{2k}\pi^{2k}}.$$

c. Prove $\zeta(2k) = \frac{2^{2k-1}\pi^{2k}}{(2k)!} B_k$.

Problem 5. Let $A, B \subset \mathbb{Z}/N\mathbb{Z}$. Define their *additive energy*

$$E(A,B) := \left|\{(a,a',b,b') \in A \times A \times B \times B : a + b = a' + b'\}\right|.$$

Prove the following two equivalent formulations.

a. For $A \subset \mathbb{Z}/N\mathbb{Z}$ and $\xi \in \mathbb{Z}/N\mathbb{Z}$, define the Fourier transform

$$\hat{\mathbf{1}}_A(\xi) = \sum_{a \in A} e^{\frac{2\pi i \xi a}{N}}.$$

Prove

$$E(A,B) = \frac{1}{N} \sum_{\xi \bmod N} \left|\hat{\mathbf{1}}_A(\xi)\hat{\mathbf{1}}_B(\xi)\right|^2.$$

b. Let $r_{A+B}(n) = \#\{a \in A, b \in B : a + b \equiv n \bmod N\}$. Then

$$E(A,B) = \sum_{n \bmod N} r_{A+B}(n)^2.$$