MATH 308, SPRING 2021 PRACTICE MIDTERM 2

MARCH 14

Each problem is worth 10 points.

Problem 1. Solve the systems:

a.

$$y'' - 3x - 2y = 0,$$
$$x'' - y + 2x = 0.$$

b.

$$x'' - y = e^t,$$
$$y'' + x = 0$$

$$x(0) = y(0) = x'(0) = y'(0) = 0.$$

Solution.

a. We have

$$\begin{pmatrix} x \\ y \end{pmatrix}'' = \begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The characteristic polynomial of the matrix is $\lambda^2 - 7$ and the eigenvalues are $\pm\sqrt{7}$ with eigenvectors $\begin{pmatrix} 1 \\ 2 \pm \sqrt{7} \end{pmatrix}$. Make the change of variables $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2+\sqrt{7} & 2-\sqrt{7} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$. Thus

$$\begin{pmatrix} u \\ v \end{pmatrix}'' = \begin{pmatrix} \sqrt{7} & 0 \\ 0 & -\sqrt{7} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Thus $u = c_1 e^{7^{\frac{1}{4}} t} + c_2 e^{-7^{\frac{1}{4}} t}$, $v = c_3 \cos\left(7^{\frac{1}{4}} t\right) + c_4 \sin\left(7^{\frac{1}{4}} t\right)$. x and y are found by inverting the linear transformation.

b. A particular solution is given by $\begin{pmatrix} x \\ y \end{pmatrix}_p = \begin{pmatrix} \frac{1}{2} e^t \\ -\frac{1}{2} e^t \end{pmatrix}$. Thus it suffices to solve the homogeneous equation. This can be solved by noting that $x'' = y$ so $x^{(4)} + x = 0$. The characteristic equation $\lambda^4 + 1 = 0$ has four roots at $\frac{1}{\sqrt{2}}(\epsilon_1 + \epsilon_2 i)$ where $\epsilon_1, \epsilon_2 = \pm 1$. Thus

$$x_h = c_1 e^{\frac{1+i}{\sqrt{2}}t} + c_2 e^{\frac{1-i}{\sqrt{2}}t} + c_3 e^{\frac{-1+i}{\sqrt{2}}t} + c_4 e^{\frac{-1-i}{\sqrt{2}}t}$$
$$y_h = i(c_1 e^{\frac{1+i}{\sqrt{2}}t} - c_2 e^{\frac{1-i}{\sqrt{2}}t} - c_3 e^{\frac{-1+i}{\sqrt{2}}t} + c_4 e^{\frac{-1-i}{\sqrt{2}}t})$$

and

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{\frac{1+i}{\sqrt{2}}t} + c_2 e^{\frac{1-i}{\sqrt{2}}t} + c_3 e^{\frac{-1+i}{\sqrt{2}}t} + c_4 e^{\frac{-1-i}{\sqrt{2}}t} + \frac{1}{2}e^t \\ i(c_1 e^{\frac{1+i}{\sqrt{2}}t} - c_2 e^{\frac{1-i}{\sqrt{2}}t} - c_3 e^{\frac{-1+i}{\sqrt{2}}t} + c_4 e^{\frac{-1-i}{\sqrt{2}}t}) - \frac{1}{2}e^t \end{pmatrix}.$$

The constants c_1, c_2, c_3, c_4 are obtained by solving the linear system

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= -\frac{1}{2} \\ c_1 - c_2 - c_3 + c_4 &= -\frac{i}{2} \\ c_1 - ic_2 + ic_3 - c_4 &= -\frac{1}{2}\frac{1-i}{\sqrt{2}} \\ c_1 + ic_2 - ic_3 - c_4 &= -\frac{i}{2}\frac{1-i}{\sqrt{2}}. \end{aligned}$$

Problem 2. Solve the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ e^t \end{pmatrix}.$$

Solution. The eigenvalues of the matrix $\begin{pmatrix} 1 & 4 \\ 0 & 5 \end{pmatrix}$ are 1 and 5 with eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. After the change of variables $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ the differential equation becomes

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u \\ 5v \end{pmatrix} + \begin{pmatrix} 1 - e^t \\ e^t \end{pmatrix}.$$

This has solution $u = c_1 e^t - te^t - 1$ and $v = c_2 e^{5t} - \frac{1}{4} e^t$.

Problem 3. Solve by Laplace transform

$$y'' + y' + y = 1, \qquad y(0) = y'(0) = 0.$$

Solution. Write $\hat{y}(s)$ for the Laplace transform. Using the initial conditions,

$$(1 + s + s^2)\hat{y}(s) = \frac{1}{s}, \qquad \hat{y}(s) = \frac{1}{s(1 + s + s^2)}.$$

Thus

$$\hat{y}(s) = \frac{A}{s} + \frac{B}{s - \frac{-1+i\sqrt{3}}{2}} + \frac{C}{s - \frac{-1-i\sqrt{3}}{2}}$$

and

$$y = A + Be^{\frac{-1+i\sqrt{3}}{2}t} + Ce^{\frac{-1-i\sqrt{3}}{2}t}$$

$$1 = A(1 + s + s^2) + Bs\left(s - \frac{-1 - i\sqrt{3}}{2}\right) + Cs\left(s - \frac{-1 + i\sqrt{3}}{2}\right).$$

so

$$A = 1, B = \frac{1}{i\sqrt{3}\left(\frac{-1+i\sqrt{3}}{2}\right)}, C = \frac{1}{-i\sqrt{3}\left(\frac{-1-i\sqrt{3}}{2}\right)}.$$

Problem 4. Show that the one dimensional equation

$$x' = \begin{cases} \sqrt{x}, & x \geq 0, \\ 0, & x < 0 \end{cases} \tag{1}$$

has infinitely many solutions. Why does this not violate the existence and uniqueness theorem?

Solution. The function $f(t) = \begin{cases} \frac{t^2}{4} & t \geq 0 \\ 0 & t < 0 \end{cases}$ has $f'(t) = \begin{cases} \frac{t}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$. The function $x(t) = f(t - c)$, for any $c \geq 0$ solves the differential equation. This does not violate the existence and uniqueness theorem because the partial derivative of $\sqrt{x}$ does not exist at 0 and is unbounded near it.

Problem 5. Show that if the largest eigenvalue of the $n \times n$ matrix A has size smaller than 1, then the power series

$$\log(I + A) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} A^k}{k}$$

is norm convergent and

$$\exp\left(\log(I + A)\right) = I + A.$$

This can be used to show that the exponential map maps a neighborhood of 0 in the Lie algebra of the special linear group one-to-one onto a neighborhood of the identity in the special linear group.

Solution. By the Jordan-Hölder theorem, A is similar to a matrix D which consists of a number of Jordan blocks all at most $n \times n$. Given a Jordan block

$$J(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & & 0 \\ & & \ddots & \ddots & \vdots \\ 0 & 0 & & & \lambda \end{pmatrix} = \lambda I + N$$

with $|\lambda| < 1$ and $N^\ell = 0$ for some $\ell \leq n$ we have $(\lambda I + N)^k = \sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j} N^j$. Thus all entries of D^k are bounded by $k^{n-1}|\lambda|^{k-n}$. It follows that $\|D^k\|_2 \leq nk^{n-1}\Lambda^{k-n}$ with $0 \leq \Lambda < 1$ the size of the largest eigenvalue. Since $A = CDC^{-1}$ implies $A^k = CD^kC^{-1}$, it follows that

$$\|A^k\|_2 \leq \|C\|_2 \|C^{-1}\|_2 n k^{n-1} \Lambda^{k-n}.$$

The series of norms now converges, so the series of matrices is norm convergent.

Since powers of A commute with norm convergent power series in A, the expression

$$\exp(\log(I + A)) = I + A$$

follows from the same formal manipulation of power series as the proof for a single variable $|x| < 1$ that $\exp(\log(1 + x)) = 1 + x$.

Problem 6. Find the equilibria of the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x(x^2+y^2-1) \\ -y(x^2+y^2+1) \end{pmatrix}$$

and discuss their stability.

Solution. Write the equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = F(x,y).$$

Equilibria occur for $F(x,y) = 0$, which occur at $(-1,0), (0,0), (1,0)$. We have

$$F'(x,y) = \begin{pmatrix} 1-3x^2-y^2 & -2xy \\ -2xy & -x^2-3y^2-1 \end{pmatrix}.$$

At $(0,0)$ one eigenvalue is 1, the other -1 so the equilibrium is unstable. At $(\pm 1, 0)$ both eigenvalues are -2, so the equilibria are stable.