MATH 308, SPRING 2021 PRACTICE MIDTERM 2

MARCH 14

Each problem is worth 10 points.

Problem 1. Solve the systems:

a.

$$y'' - 3x - 2y = 0,$$
$$x'' - y + 2x = 0.$$

b.

$$x'' - y = e^t,$$
$$y'' + x = 0$$

$$x(0) = y(0) = x'(0) = y'(0) = 0.$$

Problem 2. Solve the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ e^t \end{pmatrix}.$$

Problem 3. Solve by Laplace transform

$$y'' + y' + y = 1, \qquad y(0) = y'(0) = 0.$$

Problem 4. Show that the one dimensional equation

$$x' = \begin{cases} \sqrt{x}, & x \geq 0, \\ 0, & x < 0 \end{cases} \tag{1}$$

has infinitely many solutions. Why does this not violate the existence and uniqueness theorem?

Problem 5. Show that if the largest eigenvalue of the $n \times n$ matrix A has size smaller than 1, then the power series

$$\log(I + A) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} A^k}{k}$$

is norm convergent and

$$\exp\left(\log(I + A)\right) = I + A.$$

This can be used to show that the exponential map maps a neighborhood of 0 in the Lie algebra of the special linear group one-to-one onto a neighborhood of the identity in the special linear group.

Problem 6. Find the equilibria of the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x(x^2+y^2-1) \\ -y(x^2+y^2+1) \end{pmatrix}$$

and discuss their stability.