MATH 308, SPRING 2021 PRACTICE MIDTERM 1

MARCH 3

Each problem is worth 10 points.

Problem 1. Solve the following differential equations.

a. $y' = \frac{e^x}{y}$.
b. $y' + y \sin x = \sin x$
c. $y' = 3y, \qquad y(0) = -1$.

Solution.

a. The equation is separable, $\int yy'dy = \int e^x dx$, or $\frac{y^2}{2} = e^x + c$.
b. The integral of $\sin x$ is $-\cos x$, so the integration factor is $M(x) = e^{-\cos x}$. Let $v = M(x)y$, so $v' = e^{-\cos x} \sin x$. Then $v(x) = e^{-\cos x} + c$. It follows that $y(x) = ce^{\cos x} + 1$.
c. The equation is integrable, with solution Ce^{3x}. The initial condition implies $C = -1$.

Problem 2. Let A be the matrix $\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$.

a. Find the dimension of the image and the null space of A.
b. Give a basis for the image, kernal and $\mathbb{R}^4/\ker(A)$.

Solution.

a. The first three columns are linearly dependent since the upper 3×3 minor is upper triangular. The fourth column is in the span of the first three, so the image is 3 dimensional and the null space is 1 dimensional.

b. A basis for the image is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. A basis for the null space is $\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$. A basis for the quotient is obtained by extending the basis for the null space to a basis for $\mathbb{R}^4$. This is satisfied by the first three standard basis vectors.

Problem 3. Calculate the characteristic polynomial of the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix}$ and determine the eigenvalues, eigenvectors, and diagonalize the matrix.

Solution. We have

$$\begin{aligned} P(\lambda) &= \det \begin{pmatrix} 1-\lambda & 2 & 0 \\ 0 & 1-\lambda & 4 \\ 0 & 1 & 1-\lambda \end{pmatrix} \\ &= (1-\lambda)\det \begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix} \\ &= (1-\lambda)(3-\lambda)(-1-\lambda). \end{aligned}$$

Thus the eigenvalues are $1, 3$ and -1. The eigenvector with eigenvalue 1 is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. The eigenvector of eigenvalue -1 is in the null space of $\begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix}$, and hence is a multiple of $\begin{pmatrix} 1 \\ -1 \\ \frac{1}{2} \end{pmatrix}$. The eigenvector of eigenvalue 3 is in the null space of $\begin{pmatrix} -2 & 2 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -2 \end{pmatrix}$ and hence is a multiple of $\begin{pmatrix} 1 \\ 1 \\ \frac{1}{2} \end{pmatrix}$. Let $C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Then

$$A = C \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix} C^{-1}.$$

Problem 4. Let $\ell^2(\mathbb{N}) = \{(a_n)_{n=1}^{\infty} : \sum_n |a_n|^2 < \infty\}$. Let L and R denote the left and right shift operators on ℓ^2,

$$L(a_n) = (a_2, a_3, a_4, ...), \qquad R(a_n) = (0, a_1, a_2, a_3, ...).$$

Prove that L and R are linear. Are the maps invertible? For each map, calculate a left inverse to the map from the quotient by the null space.

Solution. Let (a_n) and (b_n) be sequences. Then $L(x(a_n) + (b_n)) = L(xa_1 + b_1, xa_2 + b_2, xa_3 + b_3, ...) = (xa_2 + b_2, xa_3 + b_3, ...) = xL(a_n) + L(b_n)$. The proof for R is similar. Thus both are linear. The map L has $(1, 0, 0, ...)$ in its null space, so is not invertible. If $R(a_n) = 0$ then $a_i = 0$ all i, so R is invertible. We have L is an inverse to R since $LR(a_n) = L(0, a_1, a_2, ...) = (a_1, a_2, ...) = (a_n)$. If $L(a_n) = 0$ then $a_i = 0$ for $i \geq 2$, and so the null space is the span of $(1, 0, 0, ...)$. The quotient by the null space forgets the first entry. The right shift is now a left inverse.

Problem 5. Let S denote the vector space of trigonometric polynomials, which is the span of the set of functions $\{e^{2\pi inx} = \cos 2\pi nx + i \sin 2\pi nx : n \in \mathbb{Z}\}$.

a. Prove that $\{e^{2\pi inx} : n \in \mathbb{Z}\}$ are linearly independent.
b. Given a trigonometric polynomial $P(x) = \sum_{k=-N}^{N} a_k e^{2\pi ikx}$, define a map M on S by $Mf(x) = P(x)f(x)$. Prove that this map is linear.
c. Let V be the subspace of S spanned by $\{e^{2\pi inx} : |n| > N\}$. Calculate the matrix of the map $M : S/V \to S/V$ given the basis $\{e^{2\pi inx} : |n| \leq N\}$.

Proof.

a. Let $f(x) = \sum_{|n|\leq N} c_n e^{2\pi inx}$. Then

$$\int_0^1 f(x)e^{-2\pi imx}dx = \int_0^1 \sum_{|n|\leq N} c_n e^{2\pi i(n-m)x}dx = c_m,$$

and hence, if the linear combination is 0, $c_m = 0$ for all m. This proves that the functions $e^{2\pi inx}$ are linearly independent.
b. If $f(x) = \sum_{|k|\leq M} b_k e^{2\pi ikx}$ and $g(x) = \sum_{|k|\leq M} c_k e^{2\pi ikx}$ then

$$\begin{aligned} M(cf(x) + g(x)) &= \left(\sum_{|n|\leq N} a_n e^{2\pi inx}\right)\left(\sum_{|k|\leq M} (cb_k + c_k)e^{2\pi ikx}\right) \\ &= \sum_{m=n+k} e^{2\pi imx}(a_n(cb_k + c_k)) \\ &= c\sum_{m=n+k} e^{2\pi mx}a_n b_k + \sum_{m=n+k} e^{2\pi mx}a_n c_k \\ &= cMf(x) + Mg(x). \end{aligned}$$

c. Put the basis in order $e^{2\pi i(-N)x}, e^{2\pi i(-N+1)x}, ..., e^{2\pi i N x}$. We have $Me^{2\pi ijx} = \sum_k a_k e^{2\pi i(j+k)x}$. It follows that the matrix is the band matrix

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots & a_{-N} & 0 & \dots & 0 \\ a_1 & a_0 & a_{-1} & \dots & a_{-N+1} & a_{-N} & \dots & 0 \\ a_2 & a_1 & a_0 & \dots & a_{-N+2} & a_{-N+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & & \vdots \\ a_N & a_{N-1} & a_{N-2} & \dots & a_0 & a_{-1} & \dots & a_{-N} \\ 0 & a_N & a_{N-1} & \dots & a_1 & a_0 & \dots & a_{-N+1} \\ \vdots & & \ddots & \ddots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_N & a_{N-1} & \dots & a_0 \end{pmatrix}.$$

$\square$

Problem 6. Let $V \subset \mathbb{R}^4$ be the set $\left\{v \in \mathbb{R}^4 : \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \cdot v = 0\right\}$. Prove that V is a subspace, determine its dimension, and give a basis.

Solution. The subspace is a null space, so is a subspace. The image is evidently 1 dimensional, so by the rank-nullity theorem, the null space is 3 dimensional. A basis is given by

$$\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$