MATH 308, SPRING 2021 PRACTICE MIDTERM 1

MARCH 3

Each problem is worth 10 points.

Problem 1. Solve the following differential equations.

a. $y' = \frac{e^x}{y}$.

b. $y' + y \sin x = \sin x$

c. $y' = 3y, \qquad y(0) = -1.$

Problem 2. Let A be the matrix $\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$.

a. Find the dimension of the image and the null space of A.

b. Give a basis for the image, kernal and $\mathbb{R}^4/\ker(A)$.

Problem 3. Calculate the characteristic polynomial of the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix}$ and determine the eigenvalues, eigenvectors, and diagonalize the matrix.

Problem 4. Let $\ell^2(\mathbb{N}) = \{(a_n)_{n=1}^{\infty} : \sum_n |a_n|^2 < \infty\}$. Let L and R denote the left and right shift operators on ℓ^2,

$$L(a_n) = (a_2, a_3, a_4, ...), \qquad R(a_n) = (0, a_1, a_2, a_3, ...).$$

Prove that L and R are linear. Are the maps invertible? For each map, calculate a left inverse to the map from the quotient by the null space.

Problem 5. Let S denote the vector space of trigonometric polynomials, which is the span of the set of functions $\{e^{2\pi inx} = \cos 2\pi nx + i \sin 2\pi nx : n \in \mathbb{Z}\}$.

a. Prove that $\{e^{2\pi inx} : n \in \mathbb{Z}\}$ are linearly independent.

b. Given a trigonometric polynomial $P(x) = \sum_{k=-N}^{N} a_k e^{2\pi ikx}$, define a map M on S by $Mf(x) = P(x)f(x)$. Prove that this map is linear.

c. Let V be the subspace of S spanned by $\{e^{2\pi inx} : |n| > N\}$. Calculate the matrix of the map $M : S/V \to S/V$ given the basis $\{e^{2\pi inx} : |n| \leq N\}$.

Problem 6. Let $V \subset \mathbb{R}^4$ be the set $\left\{v \in \mathbb{R}^4 : \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \cdot v = 0\right\}$. Prove that V is a subspace, determine its dimension, and give a basis.