MATH 308, SPRING 2021 MIDTERM 2

MARCH 14

Each problem is worth 10 points.

Problem 1.

a. Find the general solution of $y''' - 3y'' + 3y' - y = 0$.
b. Find the general solution of $y''' - 3y'' + 3y' - y = e^{2t}$.
c. Suppose $mx'' + kx' + hx$ describes the motion of a particle with mass m, force h and friction coefficient k all positive. Find the general solution of motion and explain the concept of critical damping k.

Solution.

a. Since $(D-1)^3 y = 0$ the general solution is $c_1 t^2 e^t + c_2 t e^t + c_3 e^t$.
b. Guess the particular solution Ae^{2t}. It follows $8A - 12A + 6A - A = 1$ so $A = 1$. Thus the general solution is

$$c_1 t^2 e^t + c_2 t e^t + c_3 e^t + e^{2t}.$$

c. The characteristic equation is $mr^2 + kr + h = 0$, which has roots $\frac{-k \pm \sqrt{k^2 - 4mh}}{2m}$. When $k^2 - 4mh \neq 0$ the general solution is

$$c_1 e^{\frac{-k + \sqrt{k^2 - 4mh}}{2m} t} + c_2 e^{\frac{-k - \sqrt{k^2 - 4mh}}{2m} t}.$$

Critical damping occurs when $k^2 = 4mh$, which has solution

$$c_1 t e^{-\frac{k}{2m} t} + c_2 e^{-\frac{k}{2m} t}.$$

The other parameters held constant, this gives the most rapid convergence to equilibrium, which explains the term critical damping.

Problem 2. Solve the following ODE by Laplace transform.

$$y'' + 2y' + 2y = \cos t, \qquad y(0) = 1, y'(0) = 1.$$

Solution. Take Laplace transforms

$$(s^2 + 2s + 2)\hat{y}(s) - (s + 3) = \frac{s}{s^2 + 1},$$

or

$$\hat{y}(s) = \frac{s}{(s^2 + 1)(s^2 + 2s + 2)} + \frac{s + 3}{s^2 + 2s + 2}.$$

This can be expanded in partial fractions

$$\begin{aligned} &\frac{s^3 + 3s^2 + 2s + 3}{(s - i)(s + i)(s + 1 + i)(s + 1 - i)} \\ &= \frac{A}{s - i} + \frac{B}{s + i} + \frac{C}{s + 1 - i} + \frac{D}{s + 1 + i}, \end{aligned}$$

or

$$\begin{aligned} &s^3 + 3s^2 + 2s + 3 \\ &= A(s + i)(s^2 + 2s + 2) + B(s - i)(s^2 + 2s + 2) \\ &+ C(s^2 + 1)(s + 1 + i) + D(s^2 + 1)(s + 1 - i). \end{aligned}$$

Thus $A = \frac{1-2i}{10}$, $B = \frac{1+2i}{10}$, $C = \frac{4-7i}{10}$, $D = \frac{4+7i}{10}$. It follows that

$$y(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t + \frac{4}{5}e^{-t}\cos t + \frac{7}{5}e^{-t}\sin t.$$

Problem 3. Show that if the entries in an $n \times n$ matrix $A(t) = (a_{ij}(t))$ are differentiable functions of a real variable t, then the derivative of $\det(A(t))$ is computed by differentiating the entries of one row of $A(t)$ at a time and adding the resulting n determinants.

Solution. We have

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A_{1\sigma(1)}(t) ... A_{n\sigma(n)}(t).$$

Using the product rule,

$$\begin{aligned}\frac{d}{dt}\det A(t) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{j=1}^{n} \frac{\frac{dA_{j\sigma(j)}(t)}{dt}}{A_{j\sigma(j)}(t)} A_{1\sigma(1)}(t) ... A_{n\sigma(n)}(t) \\ &= \sum_{j=1}^{n} \det \begin{pmatrix} \alpha_1(t) \\ \vdots \\ \alpha_j'(t) \\ \vdots \\ \alpha_n(t) \end{pmatrix}\end{aligned}$$

where $\alpha_1, ..., \alpha_n$ are the rows of A. This proves the claim.

Problem 4. Solve the following systems of ODEs.

a.

$$x'' - 3x - 2y'' = 0,$$
$$x'' - y'' + 2x = 0.$$

b.

$$x'' - x + y' + y = 0,$$
$$x' - x + y'' + y = 0.$$

Solution.

a. The equation in x solves $x'' + 7x = 0$, or $x = c_1 \cos(\sqrt{7}t) + c_2 \sin(\sqrt{7}t)$. Now $y'' = 2x - x'' = -5c_1 \cos(\sqrt{7}t) - 5c_2 \sin(\sqrt{7}t)$ or

$$y = \frac{5}{7}c_1 \cos(\sqrt{7}t) + \frac{5}{7}c_2 \sin(\sqrt{7}t) + c_3 t + c_4.$$

b. Let $u = x + y$, $v = x - y$. Then v satisfies $v'' - v' = 0$ or $v = c_1 e^t + c_2$. We now have

$$u'' + u' = 2v = 2(c_1 e^t + c_2)$$

or $u = c_1 e^t + 2c_2 t + c_3 e^{-t} + c_4$.

Problem 5. Write the van der Pol equation $x'' + \alpha(x^2 - 1)x' + x = 0$ as

$$x' = y,$$
$$y' = -x - \alpha(x^2 - 1)y.$$

Find the linearization near $(0, 0)$ and discuss the behavior there.

Solution. Write $\begin{pmatrix} x \\ y \end{pmatrix}' = F\begin{pmatrix} x \\ y \end{pmatrix}$. Then $F' = \begin{pmatrix} 0 & 1 \\ -1 - 2\alpha xy & -\alpha x^2 + \alpha \end{pmatrix}$. To be at equilibrium, $x = y = 0$. Here $F'(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix}$. The eigenvalues are $\frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}$. If $\alpha > 0$ then the real part of at least one eigenvalue is positive so the equilibrium is unstable. If $-2 < \alpha < 0$ then the real part is negative. If $\alpha \leq -2$ then $|\alpha| > \sqrt{\alpha^2 - 4}$ so the real part of both eigenvalues is negative. In either case the equilibrium is asymptotically stable. When $\alpha = 0$ the eigenvalues have 0 real part, so the stability is not determined by the theorem from lecture.

Problem 6. Solve the initial value problem

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

Solution. The homogeneous equation has eigenvalues $2 \pm i$ so set $x_h = c_1 e^{2t} \cos t + c_2 e^{2t} \sin t$, $y_h = c_3 e^{2t} \cos t + c_4 e^{2t} \sin t$. The matrix equation implies $c_1 = c_4, c_2 = -c_3$. This gives the homogeneous equation

$$\begin{pmatrix} x_h \\ y_h \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} \cos t + c_2 e^{2t} \sin t \\ -c_2 e^{2t} \cos t + c_1 e^{2t} \sin t \end{pmatrix}.$$

We can guess a particular solution $\begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} Ae^{2t} \\ Be^{2t} \end{pmatrix}$. Then the matrix equation becomes

$$\begin{pmatrix} 2A \\ 2B \end{pmatrix} = \begin{pmatrix} 2A - B + 1 \\ A + 2B + 2 \end{pmatrix}$$

or $A = -2$, $B = 1$. Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} \cos t + c_2 e^{2t} \sin t - 2e^{2t} \\ -c_2 e^{2t} \cos t + c_1 e^{2t} \sin t + e^{2t} \end{pmatrix}.$$

Plugging in the initial condition $c_1 = 1$, $c_2 = 3$.