MATH 308, SPRING 2021 MIDTERM 1 SOLUTIONS

MARCH 3

Each problem is worth 10 points.

Problem 1. Solve the following differential equations.

a. $y' = \frac{y}{1+x^2}$.
b. $y' + xy = x$
c. $y' = xy, \qquad y(0) = 1$.

Solution.

a. The equation is separable,

$$\int \frac{y'}{y} dy = \int \frac{dx}{1+x^2}$$

or $\ln|y| = \arctan x + C$ or $y = Ae^{\arctan x}$.

b. The integration factor is $M(x) = e^{\frac{x^2}{2}}$. Let $v(x) = M(x)y(x)$ so $v'(x) = e^{\frac{x^2}{2}} x$ so $v(x) = e^{\frac{x^2}{2}} + c$ or $y(x) = 1 + ce^{-\frac{x^2}{2}}$.

c. The equation is separable,

$$\int \frac{y'}{y} dy = \int x dx$$

thus $\ln|y| = \frac{x^2}{2} + c$ or $y = Ae^{\frac{x^2}{2}}$. The initial condition implies $A = 1$.

Problem 2. Let $A = \begin{pmatrix} 1 & 3 & 2 & 5 \\ 1 & 0 & -1 & 2 \\ 2 & 4 & 2 & 8 \\ 1 & 1 & 0 & 3 \end{pmatrix}$.

a. Find the dimension of the image and kernel of A.
b. Give a basis for the image and kernel of A and for $\mathbb{R}^4/\ker(A)$.

Solution.

a. The first two columns are linearly independent. The vectors $\begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ are in the null space. By the rank-nullity theorem, the sums of the dimensions are 4, hence both are 2.
b. The image is spanned by the first two columns, which are a basis. The null space is spanned by the two vectors above, which are a basis. A basis for the quotient by the kernel is found by extending these to a basis for $\mathbb{R}^4$. The first two standard basis vectors suffice.

Problem 3. Let V denote the vector space of binary cubic forms in two variables x, y,

$$V = \{f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 : a,b,c,d \in \mathbb{R}\}.$$

For a matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, and $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$ let

$$Tf(x,y) = f((x,y)M).$$

Calculate the 4×4 matrix of T in the basis $\{x^3, x^2y, xy^2, y^3\}$.

Solution. We have

$$\begin{aligned} Tf(x,y) &= f(\alpha x + \gamma y, \beta x + \delta y) \\ &= a(\alpha x + \gamma y)^3 + b(\alpha x + \gamma y)^2(\beta x + \delta y) \\ &+ c(\alpha x + \gamma y)(\beta x + \delta y)^2 + d(\beta x + \delta y)^3 \\ &= (a\alpha^3 + b\alpha^2\beta + c\alpha\beta^2 + d\beta^3)x^3 \\ &+ (a(3\alpha^2\gamma) + b(\alpha^2\delta + 2\alpha\beta\gamma) + c(2\alpha\beta\delta + \beta^2\gamma) + d(3\beta^2\delta))x^2y \\ &+ (a(3\alpha\gamma^2) + b(2\alpha\gamma\delta + \beta\gamma^2) + c(\alpha\delta^2 + 2\beta\gamma\delta) + d(3\beta\delta^2))xy^2 \\ &+ (a\gamma^3 + b\gamma^2\delta + c\gamma\delta^2 + d\delta^3)y^3. \end{aligned}$$

Thus the matrix is given by

$$\begin{pmatrix} \alpha^3 & \alpha^2\beta & \alpha\beta^2 & \beta^3 \\ 3\alpha^2\gamma & \alpha^2\delta + 2\alpha\beta\gamma & 2\alpha\beta\delta + \beta^2\gamma & 3\beta^2\delta \\ 3\alpha\gamma^2 & 2\alpha\gamma\delta + \beta\gamma^2 & \alpha\delta^2 + 2\beta\gamma\delta & 3\beta\delta^2 \\ \gamma^3 & \gamma^2\delta & \gamma\delta^2 & \delta^3 \end{pmatrix}.$$

Problem 4. Given the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 3 & 1 \\ 3 & 1 & 0 \end{pmatrix}$.

a. Calculate the eigenvectors and eigenvalues of A.
b. Find a matrix C such that $A = CDC^{-1}$ where D is diagonal.
c. Calculate A^{1000}.

Solution.

a. We have

$$\begin{aligned} P(\lambda) &= \det \begin{pmatrix} 1-\lambda & 0 & 3 \\ 0 & 3-\lambda & 1 \\ 3 & 1 & -\lambda \end{pmatrix} \\ &= -\lambda^3 + 4\lambda^2 + 7\lambda - 28 \\ &= -(\lambda - 4)(\lambda - \sqrt{7})(\lambda + \sqrt{7}). \end{aligned}$$

Thus the eigenvalues are $4, \pm\sqrt{7}$. The eigenvector with eigenvalue 4 is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. The eigenvector with eigenvalue $\sqrt{7}$ is in the null space of $\begin{pmatrix} 1-\sqrt{7} & 0 & 3 \\ 0 & 3-\sqrt{7} & 1 \\ 3 & 1 & -\sqrt{7} \end{pmatrix}$, and hence is $\begin{pmatrix} \frac{1+\sqrt{7}}{2} \\ -\frac{3+\sqrt{7}}{2} \\ 1 \end{pmatrix}$. The eigenvector with eigenvalue $-\sqrt{7}$ is $\begin{pmatrix} \frac{1-\sqrt{7}}{2} \\ -\frac{3-\sqrt{7}}{2} \\ 1 \end{pmatrix}$.

b. Let $C = \begin{pmatrix} 1 & \frac{1+\sqrt{7}}{2} & \frac{1-\sqrt{7}}{2} \\ 1 & -\frac{3+\sqrt{7}}{2} & -\frac{3-\sqrt{7}}{2} \\ 1 & 1 & 1 \end{pmatrix}$. Then $C^{-1}AC = \begin{pmatrix} 4 & 0 & 0 \\ 0 & \sqrt{7} & 0 \\ 0 & 0 & -\sqrt{7} \end{pmatrix}$.

c. We have $A^{1000} = C \begin{pmatrix} 4^{1000} & 0 & 0 \\ 0 & 7^{500} & 0 \\ 0 & 0 & 7^{500} \end{pmatrix} C^{-1} = 7^{500} I + (4^{1000} - 7^{500}) P_{v_1}$

where $P_{v_1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is orthogonal projection onto the eigenspace of the first eigenvector.

Problem 5.

a. Find a general formula for $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^n$.

b. Using the binomial formula $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$, find a general formula for $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^n$.

Solution. Let $M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

a. We have $M^0 = I$, $M^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $M^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $M^3 = 0$. Thus $M^n = 0$ for $n \geq 3$.

b. Since M and λI commute, $(\lambda I + M)^n = \sum_{k=0}^{n} \binom{n}{k} M^k (\lambda I)^{n-k}$. Only the $k = 0, 1, 2$ terms survive, giving

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

Problem 6. Let P_3 be the vector space of polynomials in x of degree less than 3. Give P_3 the inner product

$$\langle f, g\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)g(x)dx.$$

Calculate an orthonormal basis for P_3.

Solution. By symmetry the even and odd polynomials are already orthogonal, so we'll find an orthonormal basis for the span of $1, x^2$ and the span of x. The vector $v_1 = 1$ is already a unit vector. We have $\langle 1, x^2\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 dx = \frac{1}{12}$. Thus $x^2 - \frac{1}{12}$ is orthogonal to 1. Its norm is

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(x^2 - \frac{1}{12}\right)^2 dx = \frac{1}{180}.$$

Thus $v_3 = 6\sqrt{5}\left(x^2 - \frac{1}{12}\right)$. The norm of x is $\frac{1}{12}$, and hence $v_2 = x\sqrt{12}$.