MATH 308, SPRING 2021 MIDTERM 1

MARCH 3

Each problem is worth 10 points.

Problem 1. Solve the following differential equations.

a. $y' = \frac{y}{1+x^2}$.

b. $y' + xy = x$

c. $y' = xy, \qquad y(0) = 1.$

Problem 2. Let $A = \begin{pmatrix} 1 & 3 & 2 & 5 \\ 1 & 0 & -1 & 2 \\ 2 & 4 & 2 & 8 \\ 1 & 1 & 0 & 3 \end{pmatrix}$.

a. Find the dimension of the image and kernel of A.

b. Give a basis for the image and kernel of A and for $\mathbb{R}^4/\ker(A)$.

Problem 3. Let V denote the vector space of binary cubic forms in two variables x, y,

$$V = \{f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 : a,b,c,d \in \mathbb{R}\}.$$

For a matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, and $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$ let

$$Tf(x,y) = f((x,y)M).$$

Calculate the 4×4 matrix of T in the basis $\{x^3, x^2y, xy^2, y^3\}$.

Problem 4. Given the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 3 & 1 \\ 3 & 1 & 0 \end{pmatrix}$.

a. Calculate the eigenvectors and eigenvalues of A.

b. Find a matrix C such that $A = CDC^{-1}$ where D is diagonal.

c. Calculate A^{1000}.

Problem 5.

a. Find a general formula for $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^n$.

b. Using the binomial formula $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$, find a general formula for $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^n$.

Problem 6. Let P_3 be the vector space of polynomials in x of degree less than 3. Give P_3 the inner product

$$\langle f, g\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)g(x)dx.$$

Calculate an orthonormal basis for P_3.