Homework 8 solutions

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Problem 1. Denote by M the supremum of f on $(0, \infty)$. Let x_n be any increasing sequence diverging to infinity. We will show that $\lim_{n\to\infty} f(x_n) = M$. First, observe that the limit on the left-hand side exists because the sequence $f(x_n)$ is increasing and bounded below. Since $f(x_n) \le M$ for all n, we have $\lim_{n\to\infty} f(x_n) \le M$. On the other hand, choose any $x \in (0, \infty)$. Since x_n diverges to infinity, for all n sufficiently large we have $x \le x_n$. This implies $f(x) \le f(x_n)$ because f is increasing. Passing to the limit $n \to \infty$ we obtain the inequality $f(x) \le \lim_{n\to\infty} f(x_n)$. This implies that $\lim_{n\to\infty} f(x_n)$ is an upper bound for f. By the definition of supremum, we have $M \le \lim_{n\to\infty} f(x_n)$. Together with the previously proved inequality, this implies that $M = \lim_{n\to\infty} f(x_n)$.

Problem 2. We have $\cos x = 1 - x^2/4 + o(x^2)$ as $x \to 0$, so

$$\cos(2^{-n}\theta) = 1 - \frac{1}{4}4^{-n}\theta^2 + o(4^{-n})$$

as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} 4^n(1 - \cos(2^{-n}\theta)) = \frac{1}{4}\theta^2 + \lim_{n\to\infty} 4^n o(4^{-n}) = \frac{1}{4}\theta^2.$$

Problem 3. We have $\sin(x) = x - x^3/6 + x^5/120 + o(x^6)$ as $x \to 0$. So

$$\sin(x - x^2) = (x - x^2) + \frac{1}{6}(x - x^2)^3 + \frac{1}{120}(x - x^2)^5 + o(x^6).$$

Consider the polynomial appearing on the right-hand side. The first two summands consist of terms of degree at most 6. As regards the last summand, all of its terms are of degree at least 7 apart from the terms

$$\frac{1}{120}(x - x^2)^5 = \frac{1}{120}x^5 + \frac{5}{120}x^6 + o(x^6).$$

So if we set

$$P(x) = (x - x^2) + \frac{1}{6}(x - x^2)^3 + \frac{1}{120}x^5 + \frac{5}{120}x^6$$

we have

$$\sin(x - x^2) = P(x) + o(x^6).$$

Problem 4. The first part is an easy application of l'Hôpital's rule. To calculate the second limit, consider first the limit

$$\lim_{x\to 0} \log\left((x+e^{2x})^{1/x}\right) = \lim_{x\to 0} \frac{\log(x+e^{2x})}{x} = \lim_{x\to 0} \frac{1+2e^{2x}}{x+e^{2x}} = 3,$$

where we have used l'Hôpital's rule (we can do it since $\lim_{x\to 0}\log(x+e^{2x}) = \lim_{x\to 0} x = 0$). Now using the continuity of the exponential function, we compute

$$\lim_{x\to 0}(x+e^{2x})^{1/x} = \lim_{x\to 0} \exp\left(\log\left((x+e^{2x})^{1/x}\right)\right) = \exp\left(\lim_{x\to 0}\log\left((x+e^{2x})^{1/x}\right)\right) = e^3.$$

Problem 5. These are standard applications of l'Hôpital's rule and Taylor expansions. For example, for (1) we have $\lim_{x\to 0}(1+x)^{1/x} = e$ by the definition of e, so both the numerator and the denominator converge to zero as $x \to 0$. Computing the derivative and applying l'Hôpital's rule, we obtain

$$\lim_{x\to 0} \frac{e-(1+x)^{1/x}}{x} = -\lim_{x\to 0}(1+x)^{1/x}\left(\frac{x-(1+x)\log(1+x)}{x^2(1+x)}\right)$$

Since $\log(1+x) = x - x^2/2 + o(x^2)$, the right-hand side is equal to

$$-\lim_{x\to 0}(1+x)^{1/x}\left(\frac{x-(1+x)(x-x^2/2+o(x^2))}{x^2(1+x)}\right) = -\lim_{x\to 0}(1+x)^{1/x}\left(\frac{-x^2/2+o(x^2)}{x^2(1+x)}\right) = \frac{e}{2}.$$

Problem 6. First observe that the relation

$$f(\lambda)(1+\log(f(\lambda))) = \lambda \tag{1}$$

implies that when $\lim_{\lambda\to\infty} f(\lambda) = \infty$. Indeed, if that was not the case there would be a sequence $\lambda_i \to \infty$ such that the sequence $f(\lambda_i)$ is bounded. That would imply that the left-hand side of (1) is bounded, whereas the right-hand side goes to infinity. The contradiction shows that $\lim_{\lambda\to\infty} f(\lambda) = \infty$. Using (1) to express $f(\lambda)/\lambda$ as $(1+\log f(\lambda))^{-1}$, we arrive at

$$\lim_{\lambda\to\infty} \frac{f(\lambda)\log\lambda}{\lambda} = \lim_{\lambda\to\infty} \frac{\log\lambda}{1+\log(f(\lambda))} = \lim_{\lambda\to\infty} \frac{\log(f(\lambda)) + \log(1+\log(f(\lambda)))}{1+\log(f(\lambda))}.$$

Now, the right-hand side contains only $f(\lambda)$. As $\lambda\to\infty$, we have $f(\lambda)\to\infty$ and therefore $\log(f(\lambda))\to\infty$. Denote $y = \log(f(\lambda))$. It follows that the limit on the right-hand side is equal to the limit

$$\lim_{y\to\infty} \frac{y+\log(1+y)}{1+y},$$

which we easily compute using l'Hôpital's rule (note that both the numerator and the denominator diverge to infinity as $y\to\infty$):

$$\lim_{y\to\infty} \frac{y+\log(1+y)}{1+y} = \lim_{y\to\infty}\left(1+\frac{1}{1+y}\right) = 1.$$