Homework 8 solutions

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Problem 1. Most of the examples are standard. (1) Substitution $u = x^2 + 1$ and partial fractions. (2) Integration by parts and substitution $u = 5x$. (3) Substitution $u = 1/x$ and integration by parts. (4) Use the double angle formula and substitution $u = 5/2 + (3/2)\cos(2x)$. (5) Use the double angle formula and integrate by parts. (6) Integrate by parts four times. (7) Factor $x^3 - x = x(x-1)(x+1)$ and use partial fractions.

Let us discuss examples (8) and (9) in detail. In (8) we decompose

$$\int \frac{x}{\sqrt{x^2+x+1}} dx = \frac{1}{2} \int \frac{2x+1}{\sqrt{x^2+x+1}} dx - \frac{1}{2} \int \frac{1}{\sqrt{x^2+x+1}} dx.$$

We compute each of the terms separately. For the first one we use the substitution $u = x^2 + x + 1$. Then $du = (2x+1)dx$ and

$$\int \frac{2x+1}{\sqrt{x^2+x+1}} dx = \int u^{-1/2} du = 2u^{1/2} + C = 2\sqrt{x^2+x+1} + C.$$

The second term involves the integral

$$\int \frac{dx}{\sqrt{x^2+x+1}} = \int \frac{dx}{\sqrt{(x+1/2)^2+3/4}} = \sqrt{\frac{4}{3}} \int \frac{dx}{\sqrt{(2\sqrt{3}x + 1/\sqrt{3})^2 + 1}},$$

which we reduce to the integral $\int du/\sqrt{u^2+1} = \mathrm{arcsinh} u$ using the substitution $u = 2\sqrt{3}x + 1/\sqrt{3}$.

Next, we compute integral (9). Denote it by I_n. We have $I_0 = 1$. Now for a fixed T integrating by parts yields

$$\int_0^T x^n e^{-x} dx = \left(-x^n e^{-x}\right)\Big|_0^T + n \int_0^T x^{n-1} e^{-x} dx = -T^n e^{-T} + n \int_0^T x^{n-1} e^{-x} dx.$$

Passing to the limit $T \to \infty$, we obtain

$$I_n = n I_{n-1},$$

which shows that $I_n = n! I_0 = n!$.

Problem 2. Let us only prove the inequality $\log(1+x) < x$ for $x > 0$ as the other ones are proved in almost the same way. Set $f(x) = x - \log(1+x)$. It is a continuously differentiable function on $(0, \infty)$. Its derivative

$$f'(x) = 1 - \frac{1}{1+x}$$

is strictly positive for $x \in (0, \infty)$. Thus, f is increasing on that interval. On the other hand, $\lim_{x\to 0^+} f(x) = 0$. We conclude that $f(x) > 0$ for all $x > 0$.

Problem 3. Denote

$$a_n = \left[\left(1+\frac{1}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{1/n}$$

and

$$S_n = \log a_n = \frac{1}{n}\sum_{i=1}^{n} \log\left(1+\frac{1}{n}\right).$$

Note that S_n is the n-th Riemann sum

$$S_n = \frac{1}{n}\sum_{i=1}^{n} f\left(a + \frac{i(b-a)}{n}\right)$$

of the function $f(x) = \log x$ on the interval $[a, b] = [1, 2]$. Since f is continuous, we have

$$\lim_{n\to\infty} S_n = \int_1^2 \log x dx = (x\log x - x)|_1^2 = 2\log 2 - 1.$$

By the continuity of the function $x \mapsto e^x$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\log a_n} = e^{\lim_{n\to\infty} S_n} = e^{2\log 2 - 1} = 4/e.$$

Problem 4. Using the Fundamental Theorem of Calculus, we compute

$$\int_{-\pi}^{\pi} e^{imx}e^{-inx}dx = \int_{-\pi}^{\pi} e^{i(m-n)x}dx = \begin{cases} 0 & \text{if } m \neq n, \\ 2\pi & \text{if } m = n. \end{cases} \tag{1}$$

On the other hand, Euler's formula implies that

$$e^{imx}e^{-inx} = \{\sin(mx)\sin(nx) + \cos(mx)\cos(nx)\} + i\{\sin(mx)\cos(nx) - \cos(mx)\sin(nx)\}.$$

Taking the real part of integral (1), we obtain

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx + \int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \begin{cases} 0 & \text{if } m \neq n, \\ 2\pi & \text{if } m = n. \end{cases}$$

On the other hand, integration by parts gives

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \frac{n}{m}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx.$$

Together with the previous computation, this yields

$$\left(1+\frac{n}{m}\right)\int_{-\pi}^{\pi}\sin(mx)\sin(nx)dx = \begin{cases} 0 & \text{if } m \neq n, \\ 2\pi & \text{if } m = n. \end{cases}$$

Which shows that the integral is zero if $n \neq m$ and π if $n = m$. The integrals involving mixed terms are computed in the same way, by looking at the imaginary part(1).

Problem 5. Suppose first that f is a step function, that is there exists a subinterval $[c, d] \subset [a, b]$ such that f is equal to one on $[c, d]$ and zero otherwise. Then

$$\int_a^b f(x)\sin(\lambda x)dx = \int_c^d \sin(\lambda x)dx = \frac{\cos(\lambda d) - \cos(\lambda c)}{\lambda}.$$

Since $|\cos x| \leq 1$ for any x, the right-hand side converges to zero as $\lambda \to \infty$. So the statement is true when f is a step function. By the linearity of integrals and limits, it will be also true when f is a finite sum of step functions. Now suppose that f is any integrable function. Let $\epsilon > 0$ be any positive number. We will show that for λ sufficiently large

$$\left|\int_a^b f(x)\sin(\lambda x)dx\right| < \epsilon.$$

We have $\underline{I}(f) = \int_a^b f = \overline{I}(f)$. By the definition of $\underline{I}(f)$ as a supremum, there exists a step function $g\colon [a, b] \to \mathbb{R}$ satisfying $g \leq f$ and

$$\int_a^b (f(x) - g(x))dx < \epsilon/2.$$

Now for λ sufficiently large we have

$$\int_a^b f(x)\sin(\lambda x)dx = \int_a^b (f(x) - g(x))\sin(\lambda x)dx + \int_a^b g(x)\sin(\lambda x)dx < \epsilon/2 + \epsilon/2 = \epsilon,$$

where we have used the statement for g (which was already proved because g is a step function) and inequality $|\sin(\lambda x)| \leq 1$. Similarly we prove the lower bound

$$-\epsilon < \int_a^b f(x)\sin(\lambda x)dx$$

by considering $\overline{I}(f)$ and a step function approximating f from above.

Problem 6. To simplify the notation, assume that the domain of f is a square $[0, 1] \times [0, 1]$ rather than a disc (the proof is the same in the general case). Define $F\colon [0, 1] \to \mathbb{R}$ by $F(x) = \min\{f(x, y) \mid y \in [0, 1]\}$. Note that this function is well defined. Indeed, for every fixed $x \in [0, 1]$ the function $y \mapsto f(x, y)$ is a continuous map from $[0, 1]$ to $\mathbb{R}$ (it follows directly from the definition of continuity for functions of two variables) and as such it attains its minimum. It remains to show that F is continuous. Choose any $x_0 \in [0, 1]$ and $\epsilon > 0$. We want to show that for x sufficiently close to x_0 we have

$$|F(x) - F(x_0)| < \epsilon.$$

Now, any continuous function is uniformly continuous, see Lecture 7. The same proof works for continuous functions of two variables. It follows that there exists $\delta > 0$ such that for all x satisfying $|x - x_0| < \delta$ and for all $y \in [0, 1]$ we have

$$|f(x, y) - f(x_0, y)| < \epsilon.$$

Now let x be as above and choose $y_x \in [0, 1]$ so that

$$F(x) = \min\{f(x, y) \mid y \in [0, 1]\} = f(x, y_x).$$

Similarly, choose $y_0 \in [0, 1]$ so that

$$F(x_0) = \min\{f(x_0, y) \mid y \in [0, 1]\} = f(x_0, y_0).$$

Since $F(x_0)$ is the minimum, we have

$$F(x_0) = f(x_0, y_0) \leq f(x_0, y_x) < f(x, y_x) + \epsilon = F(x) + \epsilon.$$

Likewise, since $F(x)$ is the minimum,

$$F(x) = f(x, y_x) \leq f(x, y_0) < f(x_0, y_0) + \epsilon = F(x_0) + \epsilon,$$

which shows that $|F(x) - F(x_0)| < \epsilon$. This shows that F is continuous. In particular, it attains its minimum on $[0, 1]$. Let x_{min} be this minimum. We want to show that $F(x_{min})$ is the minimal value of f over the square $[0, 1] \times [0, 1]$. Indeed, by the definition of x_{min} and F, for any x and y in $[0, 1]$ we have

$$F(x_{min}) \leq F(x) \leq f(x, y).$$