Homework solutions

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Problem 1. (a) Proof by contradiction. Suppose that $a > b$. Choose a number ϵ satisfying $0 < \epsilon < a - b$. Since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, there exists N large so that for all $n \geq N$ we have

$$|a - a_n| < \epsilon/2, \qquad |b - b_n| < \epsilon/2.$$

Then

$$a < a_n + \epsilon/2 \leq b_n + \epsilon/2 \leq b + \epsilon,$$

which contradicts the inequality $\epsilon < a - b$.

(b) Define the positive and negative parts of f by $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \min\{f(x), 0\}$. Then $f^+ \geq 0$ and $f^- \leq 0$ everywhere. Moreover, we have

$$f(x) = f^+(x) + f^-(x), \qquad \text{and} \qquad |f(x)| = f^+(x) - f^-(x).$$

(Consider two alternatives: $f(x) \geq 0$ or $f(x) < 0$ and you will see it easily.) The functions f^+ and f^- are Riemann integrable because f is Riemann integrable (prove it). The triangle inequality gives us

$$\left| \int_a^b f(x)dx \right| = \left| \int_a^b f^+(x)dx + \int_a^b f^-(x)dx \right| \leq \left| \int_a^b f^+(x)dx \right| + \left| \int_a^b f^-(x)dx \right|$$

Since $f^+(x) \geq 0$ and $f^-(x) \leq 0$ for all x, the right-hand side is equal to

$$\int_a^b f^+(x)dx - \int_a^b f^-(x)dx = \int_a^b (f^+(x) - f^-(x))dx = \int_a^b |f(x)|dx,$$

which proves the inequality. An alternative solution: prove the inequality in question in the case when f is a step function. Passing to supremum conclude the inequality for an arbitrary Riemann integrable function.

Problem 2. Let $g \colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $g(x) = 0$ for $x \notin [0, 1]$ and $\int_{\mathbb{R}} g(x)dx = 1$. For example we consider the "triangular function"

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } x \in (0, 1/2], \\ 1 - x & \text{if } x \in (1/2, 1], \\ 0 & \text{if } x > 1. \end{cases}$$

Draw the graph of g to see that it indeed has the properties listed above. Now define $f \colon \mathbb{R} \to \mathbb{R}$ by the formula

$$f(x) = \int_0^x g(t)dt.$$

By the Fundamental Theorem of Calculus, f is differentiable. We easily check that $f(x) = 0$ for $x \le 0$ and $f(x) = 1$ for $x \ge 1$ and f is strictly increasing on $(0,1)$.

Problem 3. It is clear that f is differentiable on $\mathbb{R} \setminus \{0\}$. We check from the definition of the derivative that $f'(0)$ exists:

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = 0,$$

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} h \sin(1/h) = 0,$$

where the last equality follows from the fact that $|\sin(1/h)| \le 1$ and $\lim_{h \to 0} h = 0$. We conclude that the limit

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

exists and is equal to zero. Thus, f is differentiable. On the other hand, for $x > 0$ we have

$$f'(x) = 2x \sin(1/x) - \cos(1/x).$$

The limit $\lim_{x \to 0^+} f'(x)$ does not exist because $\lim_{x \to 0^+} 2x \sin(1/x) = 0$ and the limit $\lim_{x \to 0^+} \cos(1/x)$ does not exist (prove it). We conclude that the derivative f' is not continuous at zero.

Consider $g(x) = x + 2f(x)$. For $x \le 0$ we have $g'(x) = 1$. For $x > 0$ we have $g'(x) = 1 + 2x \sin(1/x) - \cos(1/x) > 0$ because $|cos(1/x)| \le 1$ and when $\cos(1/x) = 1$ we have $2x \sin(1/x) > 0$ (check this). So $g'(x) > 0$ as desired. On the other hand, g is not increasing in any open interval about zero. Indeed, in any such interval it attains the value 1 infinitely many times, namely at all x of the form $1/k\pi$ where $k \in \mathbb{N}$.

Problem 4. Proof by induction with respect to n. For $n = 1$ we have equality. Suppose that the inequality holds for some $n \ge 1$, so that $(1 + x)^n \ge 1 + nx$ for all $x > -1$. Multiplying both sides by $(1 + x)$, which is positive, we obtain

$$(1 + x)^{n+1} \ge (1 + nx)(1 + x) = 1 + (n + 1)x + nx^2 \ge 1 + (n + 1)x,$$

which shows that the inequality holds for $n + 1$. By the induction principle, it holds for all n. Here is an alternative solution. Set $f(x) = (1 + x)^n - 1 - nx$. We want to show that $f(x) \ge 0$ for all $x \ge -1$. It follows immediately from the binomial formula that $f(x) \ge 0$ for $x > 0$ so we need to prove the inequality for $x \in [-1, 0]$. But f is a continuous function on that interval and since $[-1, 0]$ is a bounded closed interval, f must attain its minimum. We easily check that the minimum is $f(0) = 0$, so for all $x \in [-1, 0]$ we have $f(x) \ge 0$.

Problem 5. Set $f(x) = \sum_{k=1}^{n}(x-a_k)^2$. We have $\lim_{x\to\pm\infty} f(x) = \infty$ so f must attain a global minimum at some $x_0 \in \mathbb{R}$ (prove it). At x_0 we have $f'(x_0) = 0$ where the derivative f' is given by

$$f'(x) = 2\sum_{k=1}^{n}(x-a_k).$$

So the condition $f'(x_0) = 0$ is equivalent to

$$x_0 = \frac{1}{n}\sum_{k=1}^{n}a_k,$$

which shows that x_0 is the arithmetic mean of $a_1, \ldots, a_n$.

Problem 6. We need to find the maximal distance between two points in an isosceles triangle of the given perimeter L. (Let's call this distance the *diameter* of the triangle.) Such a triangle is specified by an angle 2α between its two sides of equal length. Here $\alpha \in [0, \pi/2]$ (we allow the triangle to degenerate to an interval for $\alpha = 0$ or $\alpha = \pi/2$). Denote by a the length of each of these sides and by b the length of the third remaining side. We have $2a + b = L$ and $b = 2a\sin\alpha$ (draw a picture to see this), so

$$2a + 2a\sin\alpha = L$$

Thus, we have $a = L/2(1+\sin\alpha)$ and $b = L\sin\alpha/(1+\sin\alpha)$. The maximal distance between two points in a triangle is the length of its longest sides. The longest side might be either a or b depending on which one is greater. Suppose that a is the longest side. The inequality $a \geq b$ is equivalent to $a \geq 2a\sin\alpha$ or $1/2 \geq \sin\alpha$, which in turn is equivalent to $\alpha \in (0, \pi/6)$. The length of the longest side, as a function of α, is therefore given by

$$F\colon [0, \pi/2] \to \mathbb{R},$$

$$F(\alpha) = \begin{cases} a = \frac{L}{2(1+\sin\alpha)} & \text{for } \alpha \in (0, \pi/6], \\ b = \frac{L\sin\alpha}{1+\sin\alpha} & \text{for } \alpha \in (\pi/6, \pi/2). \end{cases}$$

The two formulae agree at $\alpha = \pi/6$ which shows that F is continuous. As a continuous function on a closed bounded interval, F must attain a maximum. First we check the endpoints. We easily verify that $F(0) = F(\pi/2) = L/2$. Now suppose that there is a maximum α_0 in the interior $(0, \pi/2)$. Since F is differentiable on $(0, \pi/6)$ and $(\pi/6, \pi/2)$, either $\alpha_0 = \pi/6$ or it lies in one of these open intervals and satisfies $F'(\alpha_0) = 0$. We compute

$$F'(\alpha) = \begin{cases} -\frac{L\cos\alpha}{2(1+\sin\alpha)^2} & \text{for } \alpha \in (0, \pi/6), \\ \frac{L\cos\alpha}{(1+\sin\alpha)^2} & \text{for } \alpha \in (\pi/6, \pi/2). \end{cases}$$

Since $\cos$ has no zeroes in $(0, \pi/2)$ such α_0 cannot exist. It remains to check the value of $F(\alpha)$ for $\alpha = \pi/6$. We have $F(\pi/6) = L/3$ which is strictly smaller than $L/2 = F(0) = F(\pi/2)$. We conclude that the greatest diameter is obtained when the triangle degenerates to a single interval of length $L/2$. Moreover, by taking triangles with $\alpha > 0$ but $\alpha \to 0$ we can obtain diameter arbitrarily close to $L/2$. This shows that the smallest circle that can cover all isosceles triangles of perimeter L has diameter equal to $L/2$.

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Bonus problem. Suppose for contradiction that f has infinitely many zeros in the interval $[0, 1]$. Perform the method of bisection to find a nested sequence of intervals $\{[a_n, b_n]\}_{n=0}^{\infty}$ such that $|b_n - a_n| = \frac{1}{2^n}$, $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for each n, and $[a_n, b_n]$ contains infinitely many zeros of f for each n. Notice that $\{a_n\}_{n=0}^{\infty}$ is an increasing sequence, which is bounded above. Let $\alpha = \sup\{a_n\}$. One may check that $a_n \to \alpha$ and $b_n \to \alpha$. Choose for each n, $x_n \in [a_n, b_n]$ such that $f(x_n) = 0$. By the squeeze principle, $x_n \to \alpha$, and thus, since f is differentiable, therefore continuous, $f(\alpha) = 0$.

Let $f'(\alpha) = c \neq 0$. Choose $\delta > 0$ sufficiently small so that, if $x \in [0, 1]$ and $|x - \alpha| < \delta$, then
$$\left| \frac{f(x) - f(\alpha)}{x - \alpha} - c \right| < \frac{|c|}{2}.$$

Since $f(\alpha) = 0$, this implies
$$\frac{|c|}{2} \leq \left| \frac{f(x)}{x - \alpha} \right|$$

so $f(x) \neq 0$. In particular, f has a single zero at α in the interval $(\alpha - \delta, \alpha + \delta)$. But $(\alpha - \delta, \alpha + \delta)$ contains $[a_n, b_n]$ for all sufficiently large n, a contradiction.