# Homework 6 solutions

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October 30, 2016

**Problem 1.** We have

$$\lim_{x\to 1}\frac{1-x^m}{1-x^n}=\lim_{x\to 1}\frac{(1-x)(1+x+\cdots+x^{m-1})}{(1-x)(1+x+\cdots+x^{n-1})}=\lim_{x\to 1}\frac{1+x+\cdots+x^{m-1}}{1+x+\cdots+x^{n-1}}=\frac{m}{n},$$

where we have used that polynomials are continuous functions.

**Problem 2.** By definition we have $\int_a^b f(x)dx = \underline{I}(f) = \overline{I}(f)$. (See the slides from Lecture 5 for definitions). Fix $\delta > 0$. Since $\underline{I}(f)$ is defined as the supremum of integrals of step functions bounded above by $f$, there is a step function $g_0$ on $[a,b]$ such that $g_0 \leq f$ and

$$\int_a^b f(x)dx - \int_a^b g_0(x)dx \leq \delta.$$

Likewise, from the definition of $\overline{I}(f)$ as an infimum, there is a step function $h_0$ on $[a,b]$ such that $f \leq h_0$ and

$$\int_a^b h_0(x)dx - \int_a^b f(x)dx \leq \delta.$$

Taking the sum of the two inequalities above, we get

$$\int_a^b h_0(x)dx - \int_a^b g_0(x)dx \leq 2\delta.$$

Moreover we have $g_0 \leq f \leq h_0$. However, the proof is not finished since $g_0$ and $h_0$ are not continuous. We claim that we can find *continuous* functions $g$ and $h$ on $[a,b]$ such that $g \leq g_0$, $h_0 \leq h$ and

$$\int_a^b g_0(x)dx - \int_a^b g(x)dx \leq \delta,$$

$$\int_a^b h(x)dx - \int_a^b h_0(x)dx \leq \delta.$$

Before proving the claim, let us see how it finishes the proof. Indeed, if such $g$ and $h$ can be found, then by adding the two inequalities above we would obtain

$$\int_a^b h(x)dx - \int_a^b g(x)dx \leq 2\delta + \int_a^b h_0(x)dx - \int_a^b g_0(x)dx \leq 4\delta.$$

Moreover we have $g \le g_0 \le f \le h_0 \le h$. Therefore, after choosing $\delta = \epsilon/4$ the functions $g$ and $h$ would have all the properties from the statement of the problem.

It remains to show that such $g$ and $h$ indeed exist. We discuss only the construction of $g$ as that of $h$ is analogous. Assume first that $[a,b] = [0,1]$ and $g_0 = u$ where $u\colon \mathbb{R} \to \mathbb{R}$ denotes the function identically equal to one on $[0,1]$ and 0 elsewhere. Then we can set $g = v$ where

$$v(x) = \begin{cases} x/\delta & \text{for } x \in [0,\delta] \\ 1 & \text{for } x \in [\delta, 1-\delta], \\ (-x+1)/\delta & \text{for } x \in [1-\delta, 1] \end{cases}$$

We easily check that $v$ is continuous. Note also that it can be extended to a continuous function on all of $\mathbb{R}$ by setting $v(x) = 0$ outside $[0,1]$. (Draw its graph to see what this function looks like!) Moreover, $v \le g_0 = 1$ and by computing areas under the graph we see that

$$\int_0^1 u(x)dx - \int_0^1 v(x)dx = 1 - (1-\delta) = \delta.$$

Now a general step function $g_0$ is of the form

$$g_0(x) = \sum_{i=1}^{N} A_i u(a_i x + b_i)$$

for some real numbers $a_i$, $b_i$, and $A_i$. Set

$$g(x) = \sum_{i=1}^{N} A_i v(a_i x + b_i)$$

where $v$ is the continuous function introduced earlier. We have $g \le g_0$ as before and

$$\int_{\mathbb{R}} g_0(x)dx - \int_{\mathbb{R}} g(x)dx = \sum_{i=1}^{N} A_i \int_{\mathbb{R}} (u(a_i x + b_i) - v(a_i x + b_i))dx = \left( \sum_{i=1}^{N} \frac{A_i}{a_i} \right) \delta,$$

where the last equality is obtained from the substitution $y = a_i x + b_i$ and the computation of $\int_0^1 (u - v)$. Thus, by making $\delta$ sufficiently small we can make $\int (g_0 - g)$ as small as we wish. This finishes the proof of the claim.

**Problem 3.** We need to show that for every $\epsilon > 0$ there exists $M$ such that for all $x > M$ we have

$$\left| \frac{1}{x} \int_0^x f(t)dt - a \right| < \epsilon. \tag{1}$$

Let us estimate the expression on the left-hand side:

$$\left| \frac{1}{x} \int_0^x f(t)dt - a \right| = \left| \frac{1}{x} \int_0^x (f(t) - a)dt \right| \le \frac{1}{x} \int_0^x |f(t) - a|dt.$$

Now $\lim_{t\to\infty} f(t) = a$ which means that for any $\delta > 0$ (to be determined later) there exists $N$ such that for $t \ge N$ we have $|f(t) - a| < \delta$. Then for any $x > N$

$$\frac{1}{x} \int_0^x |f(t) - a|dt = \frac{1}{x} \int_0^N |f(t) - a|dt + \frac{1}{x} \int_N^x |f(t) - a|dt \le \frac{\sup_{t\in[0,N]} |f(t) - a|}{x} + \frac{x - N}{x}\delta.$$

Since $1/x \to 0$ and $(x-N)/x \to 1$ as $x \to \infty$, keeping $\delta$ and $N$ fixed we can find $M$ large enough so that for all $x > M$ the right-hand side of the inequality above is not greater than $\frac{\epsilon}{2} + 2\delta$, say. If we specify $\delta$ at the beginning of the proof so that $\delta < \epsilon/4$, the number $\frac{\epsilon}{2} + 2\delta$ is smaller than $\epsilon$, which proves the estimate (1).

**Problem 4.** As explained in class, a function $f$ is convex if and only if for all $x_1$ and $x_2$ the expression

$$R(x_1, x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

is a non-decreasing function of $x_2$ when $x_1$ is fixed or vice versa. (See also `https://en.wikipedia.org/wiki/Convex_function`) Fix $x_1$ in the interior of the domain of $f$ and some interval $(a, b)$ containing $x_1$. Choose any $x_2 \in (a, b)$. Without loss of generality assume that $x_1 < x_2$. Then we have

$$R(x_1, a) \le R(x_1, x_2) \le R(x_1, b),$$

or equivalently

$$(x_2 - x_1)R(x_1, a) \le f(x_2) - f(x_1) \le (x_2 - x_1)R(x_1, b).$$

Now when $x_1$ is fixed and $x_2$ converges to $x_1$ from the right, the left-most and right-most sides of the inequality converge to zero. By the squeezing principle for limits, we have therefore

$$\lim_{x_2 \to x_1^+} f(x_2) = f(x_1).$$

Similarly we prove that $\lim_{x_2 \to x_1^-} f(x_2) = f(x_1)$, which proves that $f$ is continuous at $x_1$. The point $x_1$ was arbitrary, so $f$ is continuous at all interior points of its domain.

**Problem 5.** First, observe that we have a homeomorphism $(0, 1) \to (1, \infty)$ given by $x \mapsto 1/x$. Its inverse is $x \mapsto 1/x$ which is continuous. Now $(1, \infty)$ and $(0, \infty)$ are homeomorphic via the translation $x \mapsto x - 1$ (whose inverse is $x \mapsto x + 1$). Finally, the intervals $(0, \infty)$ and $(-\infty, \infty)$ are homeomorphic via logarithm $x \mapsto \log x$ whose inverse is $x \mapsto e^x$, both continuous functions. Since the composition of two homeomorphisms is also a homeomorphisms, composing the three functions

On the other hand, there is no homeomorphism $(0, 1) \to [0, 1]$. Indeed, the inverse of such a homeomorphism would be a continuous bijection $f\colon [0, 1] \to (0, 1)$. But any continuous function on $[0, 1]$ attains its absolute maximum. Let $a \in [0, 1]$ be such a maximum, so that for every $x \in [0, 1]$ we have

$$f(x) \le \sup_{[0,1]} f = f(a).$$

Since $f(a) \in (0, 1)$ we have $f(a) < 1$ and so $f(x) < 1$ for all $x \in [0, 1]$. This shows that no point in the non-empty open interval $(f(a), 1)$ lies in the image of $f$, which contradicts the assumption that $f$ is bijective (and thus in particular surjective).

**Bonus problem 1.** Write $f(x) = a_n \cos nx + a_{n-1}\cos(n-1)x + ... + a_0$. Notice that $\cos nx$ takes value 1 at each point of the set $\left\{\frac{2\pi k}{n}\right\}_{k=0}^{n}$. For $x$ in this set,

$$f(x) \geq a_n - (|a_{n-1}| + ... + |a_0|) > 0.$$

Notice also that $\cos nx$ takes value $-1$ at $\left\{\frac{2\pi(k+1/2)}{n}\right\}_{k=0}^{n-1}$, and thus for $x$ in this set

$$f(x) \leq -a_n + (|a_{n-1}| + ... + |a_0|) < 0.$$

Since $f$ is continuous, it changes sign twice, and hence has two zeros in each interval $\left(\frac{k\pi}{n}, \frac{(k+1)\pi}{n}\right)$, $k = 0, 1, 2, ..., n-1$, which guarantees at least $2n$ zeros.

**Bonus problem 2.** For each $\theta \in \mathbb{R}$, define $(x(\theta), y(\theta))$ to be the point on the boundary of the equilateral triangle which intersects the ray of angle $\theta$ from the origin. For example $(x(0), y(0)) = (1, 0)$. Let $T(\theta) = \sqrt{x(\theta)^2 + y(\theta)^2}$ for the radial distance. On the segment of the boundary of the triangle connecting $(1, 0)$ and $(-1/2, \sqrt{3}/2)$, that is, $0 \leq \theta \leq \frac{2\pi}{3}$, $T(\theta)$ may be determined as follows. The triangle with vertices $A = (0, 0)$, $B = (1, 0)$ and $C = (x(\theta), y(\theta))$ has angles $\theta, \frac{\pi}{6}$ and $\frac{5\pi}{6} - \theta$. Thus by sin laws,

$$\frac{T(\theta)}{\sin \pi/6} = \frac{1}{\sin(5\pi/6 - \theta)}, \quad \Leftrightarrow \quad T(\theta) = \frac{1}{2\sin(5\pi/6 - \theta)}.$$

Similar expressions may be given on the remaining arcs. One easily checks that $T(\theta)$ is continuous by connecting its values at 0, $2\pi/3$ and $4\pi/3$ from the left and right. The maximum of $T(\theta)$ is 1, and the minimum is $\frac{1}{2}$.

Define a map $f$ from the disc to the triangle by $f((0, 0)) = (0, 0)$ and, in polar coordinates, $f(r, \theta) = (T(\theta)r, \theta)$ for $r > 0$. This map is linear on rays from the origin and is evidently invertible, since $T(\theta) \neq 0$ so that the inverse map is given by $(r, \theta) \mapsto \left(\frac{r}{T(\theta)}, \theta\right)$ and $(0, 0) \mapsto (0, 0)$. Continuity at the origin holds since $T(\theta)$ is bounded above and below, so that the distance of a point to the origin changes by at most a constant factor under $f$ or its inverse. Continuity at points other than the origin holds since $T(\theta), \frac{1}{T(\theta)}$ are continuous and the transition between polar and cartesian coordinates is continuous (given by the trig functions).

**Bonus problem 3.** One can construct a homeomorphism between the triangle and the square in much the same way that the homeomorphism was constructed in the previous bonus problem, by centering each on the origin and dilating along rays emanating from the origin. We omit the details. Observe that the map

$$F(x, y) = (\cos(\sin y), \sin(\cos x))$$

defines a continuous mapping from $[0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$, and thus has a fixed point, by the Brouwer Fixed Point Theorem. This solution is in fact the unique one, as we now verify. [Note: one could avoid using the Brouwer Fixed Point Theorem for this problem, since the problem can be reduced to a single variable equation by substituting $y = \sin(\cos x)$ and solving $x = \cos(\sin(\sin(\cos x)))$.]

Define a sequence $\{(X_n, Y_n)\}_{n=0}^{\infty}$ where each member of the sequence is a pair of intervals defined by $(X_0, Y_0) = ((0, \infty), (0, \infty))$ and, for $n \geq 0$, $(X_{n+1}, Y_{n+1})$ is given by

$$X_{n+1} = \{\cos(\sin y) : y \in Y_n\}, \qquad Y_{n+1} = \{\sin(\cos x) : x \in X_n\}.$$

The fact that each $X_n$, $Y_n$ is an interval follows from the continuity of sin and cos. Let $X = \bigcap_{n=0}^{\infty} X_n$ and $Y = \bigcap_{n=0}^{\infty} Y_n$. The set of all fixed points is contained in $X \times Y$. Notice that both $X$ and $Y$ are intervals. This follows, since, if $x_1, x_2 \in X$, then $x_1, x_2 \in X_n$ for all $n$, whence $(x_1, x_2) \subset X_n$ all $n$, so $(x_1, x_2) \subset X$, and similarly $Y$.

Calculate explicitly, using $1 < \frac{\pi}{2}$ that

$$X_1 = [\cos 1, 1], \qquad Y_1 = [-\sin 1, \sin 1],$$

and, using that cos is even and decreasing on $[0, 1]$ and sin is odd and increasing on $[-1, 1]$,

$$X_2 = [\cos(\sin(\sin 1)), 1], \qquad Y_2 = [\sin(\cos 1), \sin(\cos(\cos 1))].$$

Thus $X_n \times Y_n \subset [0, 1] \times [0, 1]$ for all $n \geq 2$.

Now as maps $[0, 1] \to [0, 1]$, both cos and sin are monotonic. It follows that if $x \in X$ then $x \in X_n$ for all $n$, and there exists unique $y \in [0, 1]$ such that $y \in Y_n$ for all $n$ and $\cos(\sin y) = x$. Similarly, if $y \in Y$ then there exists $x \in X$ such that $y = \sin(\cos x)$. Thus

$$X \subset X' = \{\cos(\sin(\sin(\cos(x')))) : x' \in X\}.$$

Notice that if $X$ has endpoints $0 \leq a < b \leq 1$ then $X'$ has endpoints at

$$\cos(\sin(\sin(\cos(a)))) < \cos(\sin(\sin(\cos(b)))).$$

Use, for $x \neq y$, $|\cos x - \cos y| < |x - y|$ and $|\sin x - \sin y| < |x - y|$ to conclude that $a = b$. Since $X$ is a single point, so is $Y$.

The set of equations

$$y = \sin\left(\frac{\pi}{2} \cos \frac{\pi}{2} x\right), \qquad x = \sin\left(\frac{\pi}{2} \cos \frac{\pi}{2} y\right)$$

has solutions $(1, 0)$, $(0, 1)$, which are easily evident. Restricting to the diagonal $x = y$ produces the map $f(x) = \sin\left(\frac{\pi}{2} \cos \frac{\pi}{2} x\right)$ which maps $[0, 1]$ to itself, and thus has a further fixed point $x_0$, so that $(x_0, x_0)$ is a solution of the original equation. There may be others, I haven't checked.