Homework 5 solutions

Aleksander Doan

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Problem 1. We use Euler's formula $e^{ix} = \cos x + i \sin x$. For example, in (8) we have

$$\frac{1-e^{\pi i/2}}{1+e^{\pi i/2}} = \frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{1-2i+i^2}{1-i^2} = -i.$$

Problem 2. You can plot the functions using `https://www.wolframalpha.com/`. Just type "polar plot" followed by the function that you want to plot. For example, "polar plot sin x". To compute the area enclosed by the graph of $f = f(\theta)$ from θ_0 to θ_1, we need to compute the integral in polar cooordinates

$$A = \int_{\theta_0}^{\theta_1} f(\theta)^2 d\theta.$$

For example, in (6) we have

$$A = \int_0^{2\pi} (1+\cos\theta)^2 d\theta = \int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta) d\theta.$$

Observe that

$$\int_0^{2\pi} 1 d\theta = 2\pi.$$

$$\int_0^{2\pi} 2\cos\theta d\theta = 2\sin(2\pi) - 2\sin(0) = 0.$$

As regards the last integral, we use the double angle formula:

$$\int_0^{2\pi} \cos^2\theta d\theta = \frac{1}{2}\int_0^{2\pi} (1+\cos(2\theta)) d\theta = \pi + \frac{1}{2}\int_0^{2\pi} \cos(2\theta) d\theta$$
$$= \pi + \frac{1}{4}\int_0^{4\pi} \cos u du = \pi + \frac{1}{4}\sin(4\pi) - \frac{1}{4}\sin(0) = \pi,$$

where in the second line we have used the substitution $u = 2\theta$. We conclude that $A = 3\pi$. The other examples are similar.

Problem 3. Let
$$f(x) = \sqrt{1+\sqrt{x}}.$$
Note that f is strictly increasing on $[0,9]$ and $f(0)=1$, $f(9)=2$, therefore there is an inverse function $g\colon [1,2] \to [0,9]$. Instead of computing the integral of f over $[0,9]$, we will compute the integral of g over $[1,2]$. The two are related by

$$\int_0^9 f(x)dx + \int_1^2 g(y)dy = 9 \times 2 = 18.$$

(Draw a picture to illustrate this!) In order to find $g = f^{-1}$ we solve $f(x) = y$ for x.

$$y = \sqrt{1+\sqrt{x}},$$
$$y^2 - 1 = \sqrt{x},$$
$$(y^2-1)^2 = x.$$

So $g(y) = (y^2-1)^2$ is the inverse of f. We compute

$$\int_1^2 g(y)dy = \int_1^2 (y^4 - 2y^2 + 1)dy = \left[\frac{1}{5}y^5 - \frac{2}{3}y^3 + y\right]_1^2 = \frac{38}{15}.$$

Therefore,
$$\int_0^9 f(x)dx = 18 - \int_1^2 g(t)dy = 18 - \frac{38}{15} = \frac{232}{15}.$$

Problem 4. The integral in question is the sum of two integrals

$$\int_{-2}^2 x\sqrt{4-x^2}dx - 3\int_{-2}^2 \sqrt{4-x^2}dx.$$

The first one is zero because the function $f(x) = x\sqrt{4-x^2}$ is odd (that is, $f(-x) = -f(x)$), and we integrate it over an interval symmetric with respect to 0. To compute the second integral, we observe that the equation of the circle of radius 2 and centre at $(0,0)$ is
$$x^2 + y^2 = 4,$$
so the graph of the function $y = \sqrt{4-x^2}$ is the upper-half circle. Its area is 2π (since the radius is 2 and the area of the entire circle is 4π), so

$$\int_{-2}^2 \sqrt{4-x^2}dx = 2\pi.$$

Using the previous computation we obtain

$$\int_{-2}^2 (x-3)\sqrt{4-x^2}dx = -6\pi.$$

Problem 5. The average value of $\sin x$ over $[0, \pi/2]$ is

$$\frac{1}{\pi/2}\int_0^{\pi/2} \sin x dx = \frac{2}{\pi}(-\cos(\pi/2) + \cos(0)) = \frac{2}{\pi}.$$

For $\sin^2 x$ we proceed in the same way as in the solution to Problem 1 (where we integrated $\cos^2 x$), using the double angle formula and substitution $u = 2x$.

Problem 6. Let us discuss only example (4) as the others are similar. The total mass of the rod is

$$m = \int_0^L \rho(x)dx = \int_0^{L/2} xdx + \int_{L/2}^L \frac{L}{2}dx = \frac{1}{2}(L/2)^2 + (L/2)^2 = \frac{3}{8}L^2.$$

The centre of mass is

$$\begin{aligned} x_c = \frac{\int_0^L x\rho(x)dx}{m} &= \frac{1}{m}\left\{\int_0^{L/2} x^2 dx + \int_{L/2}^L \frac{L}{2}xdx\right\} \\ &= \frac{1}{m}\left\{\frac{1}{3}(L/2)^3 + \frac{L}{4}(L^2 - (L/2)^2)\right\} = \frac{11/48L^3}{3/8L^2} = \frac{11}{18}L. \end{aligned}$$

The moment of inertia is

$$\begin{aligned} I = \int_0^L x^2\rho(x)dx &= \int_0^{L/2} x^3 dx + \int_{L/2}^L \frac{L}{2}x^2 dx \\ &= \frac{1}{4}(L/2)^4 + \frac{L}{6}(L^3 - (L/2)^3) = \frac{31}{192}L^4. \end{aligned}$$

Finally, the radius of gyration is

$$r^2 = \frac{I}{m} = \frac{31}{72}L^2.$$

Problem 7. Consider the function $g\colon [a,b] \to \mathbb{R}$ given by $g(x) = f(x) - x$. It is continuous as the sum of two continuous functions. We have

$$g(a) = f(a) - a \geq 0 \qquad \text{and} \qquad g(b) = f(b) - b \leq 0$$

because $f(a)$ and $f(b)$ belong to the interval $[a,b]$. By the intermediate value theorem, there exists $c \in [a,b]$ such that $g(c) = 0$. But this is equivalent to $f(c) = c$.

Problem 8. Let x be any real number. For every natural number n the open interval $(x - \frac{1}{n}, x + \frac{1}{n})$ contains a point of A because A is dense. Choose any such point and call it x_n. Since $|x - x_n| < \frac{1}{n}$, we have $\lim_{n\to\infty} x_n = x$. Since all of x_n belong to A, we have $f(x_n) = 0$ for all n. On the other hand, f is continuous at x, so

$$f(x) = \lim_{n\to\infty} f(x_n) = 0.$$

Since x was chosen arbitrarily, we have $f = 0$ everywhere.

Bonus problem 1. Let $\alpha = \sup\{x \in [0,1] : f(x) > 0\}$. We claim $f(\alpha) = 0$. Suppose otherwise. Since g is continuous at α, there is $\delta > 0$ such that $x \in [0,1]$ and $|x - \alpha| < \delta$ implies $|g(x) - g(\alpha)| < \frac{|f(\alpha)|}{2}$. Suppose first that $f(\alpha) > 0$. Then $\alpha < 1$. It follows that for some $\alpha < x \le \min(\alpha + \delta, 1)$, $f(x) \le 0$, and thus

$$f(x) + g(x) \le g(x) \le g(\alpha) + |g(x) - g(\alpha)| \le g(\alpha) + \frac{f(\alpha)}{2} < g(\alpha) + f(\alpha).$$

This contradicts the fact that $f + g$ is non-decreasing. Suppose instead that $f(\alpha) < 0$. Then $\alpha > 0$ and so (by the supremum property) there exists x, $\max(0, \alpha - \delta) < x < \alpha$ for which $f(x) > 0$. Check that

$$f(x) + g(x) \ge g(x) \ge g(\alpha) - |g(x) - g(\alpha)| \ge g(\alpha) + \frac{f(\alpha)}{2} > g(\alpha) + f(\alpha)$$

which again contradicts the fact that $f + g$ is non-decreasing.

Bonus problem 2. As a lower step function, take the constant function 0, which has integral 0. Note that for each $n \ge 1$ there are no more than n^2 reduced fractions $\frac{p}{q}$, $0 \le p < q$ with denominator at most n. Define upper step functions $s_n(x)$ which take value 1 if $\left|x - \frac{p}{q}\right| < \frac{1}{2n^3}$ for some $1 \le q \le n$, $0 \le p < q$, $s_n(1) = 1$ and take value $\frac{1}{n}$ otherwise. Notice that $g(x) \le s_n(x)$ for each n. Also,

$$\int_0^1 s_n(x)dx \le \int_0^1 \frac{dx}{n} + \sum_{1 \le q \le n} \sum_{0 \le p < q} \int_{\frac{p}{q} - \frac{1}{2n^3}}^{\frac{p}{q} + \frac{1}{2n^3}} dx \le \frac{1}{n} + \frac{n^2}{n^3} = \frac{2}{n}.$$

Letting $n \to \infty$ proves that g is integrable with integral 0.