Homework 4 solutions

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Problem 1. Suppose by contradiction that $x > 0$. Then $x/2 > 0$ and setting $h = x/2$ we would get $x < x/2$ or equivalently $1 < 1/2$, which is a contradiction. That shows that $x \leq 0$ and since we also have $0 \leq x$ it follows that $x = 0$.

Problem 2. By the Cauchy-Schwarz inequality

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right) \geq \left(\sum_{k=1}^{n} \sqrt{x_k}\sqrt{y_k}\right)^2 = \left(\sum_{k=1}^{n} \sqrt{x_k}\frac{1}{\sqrt{x_k}}\right)^2 = \left(\sum_{k=1}^{n} 1\right)^2 = n^2.$$

Problem 3. Write $w = a + bi$ and $z = c + di$. We need to prove that

$$L = |w + z|^2 \leq (|w| + |z|)^2 = |w|^2 + |z|^2 + 2|w||z| = R.$$

The left-hand side is

$$L = (a + b)^2 + (c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2(ab + cd),$$

whereas the right-hand side is

$$R = a^2 + b^2 + c^2 + d^2 + 2\sqrt{(a^2 + b^2)(c^2 + d^2)}$$

so the inequality $L \leq R$ is equivalent to

$$ab + cd \leq \sqrt{(a^2 + b^2)(c^2 + d^2)},$$

which is the Cauchy-Schwarz inequality.

Problem 4. By the invariance under translation (see Lecture 5, slides 31-32) we have

$$\int_a^b f(x)dx = \int_0^{b-a} f(a+x)dx.$$

On the other hand, rescaling the interval $[0, b-a]$ by $k = 1/(b-a)$ we get

$$\int_0^{b-a} f(a+x)dx = \frac{1}{k}\int_0^{k(b-a)} f\left(a+\frac{x}{k}\right)dx = (b-a)\int_0^1 f(a+(b-a)x)dx$$

as we wanted to prove.

Problem 5. (1) First observe that for any x we have

$$1-x^p = (1-x)+(x-x^2)+(x^2-x^3)+\cdots+(x^{p-1}-x^p) = (1-x)(1+x+x^2+\cdots+x^{p-1}).$$

Suppose that $b \neq 0$ and set $x = a/b$ in the formula above to obtain

$$1-\frac{a^p}{b^p} = \left(1-\frac{a}{b}\right)\left(1+\frac{a}{b}+\cdots+\frac{a^{p-1}}{b^{p-1}}\right).$$

Multiplying both sides by b^p (which on the right-hand side we distribute by multiplying the first bracket by b and the second by b^{p-1}) yields

$$b^p - a^p = (b-a)(b^{p-1}+b^{p-2}a+\cdots+a^{p-1})$$

as desired. We have proved the formula under the assumption that $b \neq 0$ but it obviously holds for $b = 0$ as well.

(2) Apply the first part to $b = n+1$ and $a = n$. The result is

$$(n+1)^p - n^p = (n+1)^{p-1} + (n+1)^{p-2}n + \cdots + (n+1)n^{p-2} + n^{p-1}.$$

There are p terms on the right-hand side. Apart from the first one, each of them is strictly smaller than $(n+1)^{p-1}$ which shows that

$$(n+1)^p - n^p < p(n+1)^{p-1}.$$

Likewise, each of the terms in the sum apart from the last one is strictly greater than n^{p-1}, which shows that

$$(n+1)^p - n^p > pn^{p-1}.$$

This proves (2) (where we have replaced $p+1$ by p).

(3) We prove the inequality by induction with respect to n. First, for $n = 1$ the inequality is

$$0 < \frac{1}{p+1} < 1,$$

which is clearly true for any positive integer p. Suppose now that the inequality holds for some $n \geq 1$. We will show that it also holds for $n+1$. From part (2) of the problem and the induction hypothesis we obtain

$$\frac{(n+1)^{p+1}}{p+1} < \frac{n^{p+1}}{p+1} + (n+1)^p < \sum_{k=1}^{n} k^p + (n+1)^p = \sum_{k=1}^{n+1} k^p.$$

On the other hand,

$$\frac{(n+1)^{p+1}}{p+1} > \frac{n^{p+1}}{p+1} + n^p = \sum_{k=1}^{n-1} k^p + n^p = \sum_{k=1}^{n} k^p,$$

which proves the statement for $n+1$. By the induction principle, the inequality holds for all n.

Bonus problem. Given an integer n, write $1_{\{n\}}$ for the indicator function of the half-open interval $[n, n+1) \subset \mathbb{R}$. Given a subset $S \subset \mathbb{Z}$, write $1_S = \sum_{n \in S} 1_{\{n\}}$. Thus, if S is a finite set contained in $[-M, M-1]$ for some $M > 0$, then $\int_{-M}^{M} 1_S(x)dx = |S|$ gives the cardinality of S. We suppose all of the finite subsets given satisfy $S_i \subset [-M, M-1]$.

We claim

$$\left| \bigcup_{i=1}^{n} S_i \right| = \int_{-M}^{M} 1 - \prod_{i=1}^{n} (1 - 1_{S_i})(x)dx. \tag{1}$$

Indeed, the function integrated on the right hand side takes value 1 on any interval $[m, m+1)$ such that $m \in \bigcup_{i=1}^{n} S_i$, and nowhere else. To obtain the desired claim, note that $1_S 1_T = 1_{S \cap T}$ and expand

$$\prod_{i=1}^{n} (1 - 1_{S_i}) = \sum_{j=0}^{n} (-1)^j \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq n} 1_{S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_j}}.$$

Exchange the order of summation and integration in (1) to complete the proof.

Bonus problem. Observe that each ζ_k satisfies $\zeta_k = e^{\frac{2\pi i k}{n}}$, and hence, $\zeta_k^n = 1$. They are all distinct, whence one obtains the factorization

$$x^n - 1 = \prod_{k=1}^{n} (x - \zeta_k)$$

(there are at most n roots on the left, and n roots have been identified). We now use a helpful algebraic fact relating the roots and coefficients of a polynomial. Set $P(x) = \prod_{j=1}^{n}(x - r_j) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_0$. The coefficients have value

$$a_k = (-1)^{n-k} \sum_{1 \leq i_1 < i_2 < ... < i_{n-k} \leq n} r_{i_1} r_{i_2} ... r_{i_{n-k}}.$$

Without the factor of $(-1)^{n-k}$, this latter sum is called the $(n-k)th$ elementary symmetric polynomial on $r_1, ..., r_k$, denoted $e_{n-k}(r_1, ..., r_n)$. Matching coefficients, one finds

$$\prod_{k=1}^{n} \zeta_k = (-1)^{n+1}, \qquad \forall 1 \leq j < n, \sum_{1 \leq i_1 < ... < i_j \leq n} \zeta_{i_1} ... \zeta_{i_j} = 0.$$

Plug in $x = -1$ to obtain

$$(-1)^n - 1 = (-1)^n \prod_{k=1}^{n} (1 + \zeta_k) \qquad \Rightarrow \qquad \prod_{k=1}^{n} (1 + \zeta_k) = 1 + (-1)^{n-1}.$$

Finally, write

$$\frac{x^n - 1}{x - 1} = \prod_{k=1}^{n-1} (x - \zeta_k) = 1 + x + x^2 + ... + x^{n-1}.$$

Evaluate this at $x = 1$ to obtain

$$\prod_{k=1}^{n-1} (1 - \zeta_k) = n.$$