Homework 3 solutions

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Problem 1. Rational numbers form a field, so in particular they are closed under addition and multiplication. Thus, if x and $x + y$ are rational, so is $y = (x + y) - x$. Likewise, if x and xy are rational and $x \neq 0$, then so is $y = (xy)x^{-1}$.

Problem 2. (Recall the definition of a Dedekind cut from Lecture 3, slide 10.) It is obvious that $\alpha+\beta$ is not empty because the same is true for α and β. Every Dedekind cut is bounded above (since otherwise it would be all of $\mathbb{Q}$) so there are rational numbers M and N such that for all $x \in \alpha$ we have $x \leq M$ and for all $y \in \beta$ we have $y \leq N$. We conclude that $x+y \leq M+N$. It follows that $\alpha+\beta$ is bounded above so $\alpha+\beta \neq \mathbb{Q}$. This proves that the first axiom is satisfied. To check the second axiom, suppose that $x \in \alpha$, $y \in \beta$ and q is a rational number satisfying $q < x+y$. We have $q-x < y$ which implies that $q-x \in \beta$ since β is a Dedekind cut. So $q = x+(q-x)$ is the sum of an element of α and an element of β and $q \in \alpha+\beta$, which proves that $\alpha+\beta$ satisfies the second axiom. It remains to prove that $\alpha+\beta$ satisfies the third axiom. Again, let $x \in \alpha$ and $y \in \beta$. Then there exist $r_x \in \alpha$ and $r_y \in \beta$ such that $x < r_x$ and $y < r_y$. The sum $r_x + r_y$ is then an element of $\alpha+\beta$ such that $x+y < r_x + r_y$, which shows that the third axiom is satisfied. Therefore, $\alpha+\beta$ is a Dedekind cut.

Problem 3. Suppose that $y - x > 1$. Let $[x]$ be the largest integer satisfying $[x] \leq x$ (prove that such an integer exist). Then $x < [x]+1$ and $[x]+1 \leq x+1 < y$, so $[x]+1$ belongs to the open interval (x, y), which proves the first part of the problem. Suppose now that x and y satisfy $x < y$. Since $y - x > 0$, there exists a natural number m such that $m(y-x) > 1$ or equivalently $my - mx > 1$. Applying the first part of the problem to the numbers my and mx, we see that there is an integer n such that $my > n > mx$. Dividing all sides by m we obtain $y > n/m > x$, which proves the second part of the problem. To prove the third part, observe that the open interval (x, y) is an uncountable set. On the other hand, rational numbers form a countable set. This means that there must exist an irrational number $z \in (x, y)$, which means that $x < z < y$.

Problem 4. Denote $\phi = (1+\sqrt{5})/2$. A direct calculation shows that ϕ solves the quadratic equation $1+\phi = \phi^2$. We prove the statement by (generalised) induction with respect to n. For $n = 2$ we have $F_2 = 1 < \phi$ because $\sqrt{5} > 1$. Suppose that the statement is true for all natural numbers smaller than or equal to some $n \geq 2$. We want to conclude that it is true for $n+1$. By the induction hypothesis

$$F_{n+1} = F_{n-1} + F_n \leq \phi^{n-2} + \phi^{n-1} = \phi^{n-2}(1+\phi) = \phi^n,$$

which shows that the inequality holds for $n+1$. By the induction principle, it holds for all $n \geq 2$.

Problem 5. Define $T_1 = 1$ and, recursively, for $n > 1$, $T_n = S_n \setminus \bigcup_{1\leq m<n} S_m$. Then the sets T_n are countable and pairwise disjoint, and $S = \bigcup_{n=1}^{\infty} S_n = \bigcup_{n=1}^{\infty} T_n$. Thus we may assume that the initial sets S_n were pairwise disjoint.

Since each of S_n is countable, there is an injective map $f_n\colon S_n \to \mathbb{N}$ for every n. We construct a map $F\colon S \to \mathbb{N}\times\mathbb{N}$ by

$$F(x) = (f_n(x), n) \qquad \text{for } x \in S_n.$$

It is well-defined since S is the disjoint union of all S_n. The map is clearly injective: if $(f_n(x), n) = (f_m(y), m)$ for some natural numbers n,m and $x \in S$, $y \in S$, then we have $n = m$, which shows that both x and y are elements of the same set S_n, and $f_n(x) = f_n(y)$, which shows that $x = y$ since f_n is injective. To sum up, we have constructed an injective map from S to $\mathbb{N}\times\mathbb{N}$. On the other hand, there is an injection $\mathbb{N}\times\mathbb{N} \to \mathbb{N}$ given by

$$g(m,n) = 2^m 3^n.$$

The injectivity of g follows from the fact that the prime decomposition of a natural number is unique. Thus, the composition $g \circ f$ gives an injective map from S to $\mathbb{N}$, proving that S is countable.

Problem 6. First, we show that the decomposition, if exists, is unique. Suppose that we have another pair of polynomials Q_1 and R_1 satisfying $\deg R < \deg B$ and

$$QB + R = P = Q_1 B + R_1.$$

Then

$$(Q - Q_1)B = R_1 - R.$$

The polynomial on the right-hand side is of degree strictly smaller than $\deg B$. On the other hand, the polynomial on the right-hand side has degree $\deg(Q - Q_1) + \deg B$. So we have

$$\deg(Q - Q_1) + \deg B = \deg(R_1 - R) < \deg B,$$

which can happen only if $\deg(Q - Q_1) < 0$ and so $Q - Q_1 = 0$. This shows that $Q = Q_1$ and $R = R_1$. This shows uniqueness of the decomposition.

It remains to prove that such Q and R exist. We prove it by (generalised) induction with respect to the degree $n = \deg P$. Denote $m = \deg B$. The statement clearly holds when $n < m$, in which case we take $Q = 0$ and $R = P$. This proves the first inductive step. Suppose that the statement holds for all natural numbers smaller than some $n \geq m$. We want to conclude that it also holds for n. Write

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n, \qquad a_n \neq 0,$$

$$B(x) = b_0 + b_1 x + \cdots + b_m x^m, \qquad b_m \neq 0.$$

Consider the polynomial

$$P_1(x) = P(x) - \frac{a_n x^{n-m}}{b_m} B(x).$$

(Note that we are using here the assumption that $n \geq m$.) The degree of P_1 is at most n. However, the coefficient in front of the power x^n in $P_1(x)$ is

$$a_n - \frac{a_n}{b_m} b_m = 0,$$

which shows that $\deg P_1 < n$. By the induction hypothesis, there are polynomials Q_1 and R_1 such that $\deg R_1 < \deg B$ and

$$P_1 = Q_1 B + R_1.$$

Equivalently,

$$P - \frac{a_n x^{n-m}}{b_m} B = Q_1 B + R_1.$$

After rearranging, we obtain

$$P = \left(Q_1 + \frac{a_n x^{n-m}}{b_m} \right) B + R_1.$$

Setting $Q = Q_1 + \frac{a_n x^{n-m}}{b_m}$ and $R = R_1$ we obtain

$$P = QB + R$$

and $\deg R < \deg B$ as desired. This shows that the decomposition exists for all polynomials P of degree n. By the induction principle, the statement is true for all natural numbers n.

Bonus problem. Let $S = [0, 1, 2, ..., 999]^3$ and let $f : S \to \mathbb{R}$ be defined by $f(n_1, n_2, n_3) = n_1 + n_2\sqrt{2} + n_3\sqrt{3}$. The range of f is contained in the interval $[0, 4147]$, and hence when it is broken into consecutive half-open intervals of length 4.2×10^{-6}, there are fewer than 10^9 such intervals. By the pigeonhole principle, two elements of S say $(m_1, m_2, m_3) \neq (m_1', m_2', m_3')$ map to the same interval. Observe that $n_1 = m_1 - m_1'$, $n_2 = m_2 - m_2'$, $n_3 = m_3 - m_3'$ are not all zero and satisfy $|n_1|, |n_2|, |n_3| < 1000$ and

$$\left| n_1 + n_2\sqrt{2} + n_3\sqrt{3} \right| < 4.2 \times 10^{-6}.$$