Homework 1 solutions

Aleksander Doan

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Problem 1. An element $x \in X$ is in $(A \cap B)^c$ if and only if $x \notin A \cap B$. This is equivalent to: $x \notin A$ or $x \notin B$, which in turn is equivalent to $x \in A^c \cup B^c$. This shows that $(A \cap B)^c = A^c \cup B^c$. The second equality is proved in a similar way.

In what follows we adopt the convention that $\mathbb{N}$ does not contain zero.

Problem 2. Fix $n \in \mathbb{N} \cup \{0\}$. From assumptions (1), (2), (3), and the induction principle it follows that $P(n+k, k)$ is true for all $k \in \mathbb{N} \cup \{0\}$. Likewise, for any $m \in \mathbb{N} \cup \{0\}$ the statement $P(k, m+k)$ is true. Now let (n, m) be arbitrary. By the trichotomy law, we have $n \leq m$ or $m \leq n$. Without loss of generality assume that the first holds. Then by definition there exists $k \in \mathbb{N} \cup \{0\}$ such that $n + k = m$. Then $P(n, m)$ is the same as $P(n, n+k)$ which we have already proved to be true.

Problem 3. (Recall: $a < b$ if and only if there exists $c \in \mathbb{N}$ such that $a + c = b$.) First we check that th three relations $m = n$, $m < n$, $m > n$ are mutually exclusive. Indeed, if $m > n$ or $m < n$ then by definition $m \neq n$. On the other hand, suppose by contradiction that $m > n$ and $n < m$. Then $m = n + c$ and $n = m + d$ for some $c, d \in \mathbb{N}$. Then we would get $m = n + c = m + c + d$ which leads to a contradiction since $c + d \neq 0$. This shows that if one alternative holds, then the other cannot hold.

For $n \in \mathbb{N} \cup \{0\}$ denote by $P(n)$ be the statement: for any $m \in \mathbb{N} \cup \{0\}$ we have $m = n$ or $m < n$ or $n < m$. We prove by induction with respect to n that $P(n)$ holds for all n. First we check that $P(0)$ is true. For every m we either have $m = 0$ or $m \neq 0$. If the latter is true, then $m > 0$ because $0 = 0 + m$. Therefore $P(0)$ is true. Now suppose that $P(n)$ holds for some n. Let $m \in \mathbb{N} \cup \{0\}$. By the induction hypothesis, there are two possibilities. First is that $m \leq n$. Then $m \leq n < n + 1$. Second is that $m > n$. Then there exists $c \in \mathbb{N}$ such that $n + c = m$. Now, since $c \neq 0$ the axiom of induction implies that $c = d+1$ for some $d \in \mathbb{N} \cup \{0\}$. Thus $(n+1)+d = m$ which shows that $m \geq n+1$. Therefore, we proved that in all cases either $m = n+1$ or $m < n+1$ or $m > n+1$, which is exactly the statement $P(n+1)$.

Problem 4. For $n \geq 1$ denote by $P(n)$ denote the equality from the problem. We prove that $P(n)$ is true by induction. First, $P(1)$ is equivalent to $2-1 = 1$ which is clearly true. Suppose that $P(n)$ is true for some $n \geq 1$. Then we get

$$(n+1)^2 = n^2 + 2n + 1 = \sum_{i=1}^{n}(2i-1) + (2(n+1)-1) = \sum_{i=1}^{n+1}(2i-1),$$

which proves that $P(n+1)$ is true.

Problem 5. First we show that such a decomposition is unique. Indeed, suppose that for some other q' and r' with $0 \leq r' < b$ we have

$$n = qb + r = q'b + r'.$$

Then

$$(q - q')b = r' - r$$

which shows that b divides $r' - r$. Since $-b < r' - r < b$ this is possible only if $r' - r = 0$ or equivalently $r' = r$. Then $(q - q')b = 0$ and since $b > 0$ we must have $q = q'$. Thus, $(q, r) = (q', r')$ and the decomposition is unique.

It remains to prove that such a decomposition exists for every positive n. We prove this by induction. First, let $n = 1$. If $b = 1$ then we take $q = 1$ and $r = 0$. If $b > 1$, then we take $q = 0$ and $r = 1$. So the statement is true for $n = 1$. Assume now that it is true for some $n \geq 1$. That is, we have

$$n = qb + r.$$

Then

$$n + 1 = qb + r + 1.$$

Since $r < b$ we have $r + 1 \leq b$. There are two cases. If $r + 1 = b$, then

$$n + 1 = qb + b = (q + 1)b$$

which gives us the desired decomposition. On the other hand, if $r + 1 < b$, then $n + 1 = qb + (r + 1)$ is already of the desired form. This proves that the statement is true for $n + 1$.

Bonus problem The proof is by induction on the number of lines.

Base case (no lines drawn): Color the Euclidean plane either red or blue. There are no adjacent regions to produce a conflict.

Inductive step: Let $n \geq 0$ and assume that for any configuration of n lines, the resulting regions may be colored so that no two adjacent regions have the same color. Now suppose that $n+1$ lines have been drawn. Hide one line ℓ and perform a legal coloring C on the regions produced by the remaining n lines. Now observe that ℓ splits Euclidean space into two half-planes P_1 and P_2, splitting some regions produced by the other n lines. In all regions that belong to P_2, reverse the color from C, while in all regions that belong to P_1 leave the color the same. In the subdivision of space produced by the $n+1$ lines, if two neighboring regions are on the same side of ℓ then they have opposite colors by the inductive assumption. If they are on opposite sides of ℓ, then to be neighbors they must have a segment of ℓ in common on their boundary. It follows that the two regions were the same region prior to drawing ℓ, hence had the same color prior to the recoloring, and now have opposite colors.