MATH 141, FALL 2016, HW4

DUE IN SECTION, SEPTEMBER 22

Problem 1. Let x be a real number such that $0 \leq x < h$ for every positive real number h. Prove that $x = 0$.

Problem 2. If $x_1, ..., x_n$ are positive real numbers and $y_k = \frac{1}{x_k}$, prove that

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right) \geq n^2.$$

Problem 3. Prove the triangle inequality for $\mathbb{C}$: if w and z are complex numbers then

$$|w + z| \leq |w| + |z|.$$

Problem 4. Let f be an integrable function on $[a, b]$. Using Theorems stated or proved in class, derive the formula

$$\int_a^b f(x)dx = (b - a)\int_0^1 f[a + (b - a)x]dx.$$

Problem 5. (1) Let p be a positive integer. Prove that

$$b^p - a^p = (b - a)(b^{p-1} + b^{p-2}a + b^{p-3}a^2 + \cdots + ba^{p-2} + a^{p-1}).$$

(2) Let p and n denote positive integers. Use the first part to show that

$$n^p < \frac{(n + 1)^{p+1} - n^{p+1}}{p + 1} < (n + 1)^p.$$

(3) Use induction to prove that

$$\sum_{k=1}^{n-1} k^p < \frac{n^{p+1}}{p + 1} < \sum_{k=1}^{n} k^p.$$

Bonus Problem. Given $S \subset \mathbb{Z}_{>0}$, denote its cardinality by $|S|$, so $|S|$ is the number of elements of S if S is finite, or $|S| = \infty$ otherwise. Let $S_1, S_2, ..., S_n \subset \mathbb{Z}_{>0}$ be finite subsets. Prove the inclusion-exclusion formula

$$\left|\bigcup_{i=1}^{n} S_i\right| = \sum_{j=1}^{n} (-1)^{j+1} \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \left|S_{i_1} \cap \cdots \cap S_{i_j}\right| .$$

Bonus Problem. Let $n \ge 2$ be an integer, and set $\zeta_k = \cos\frac{2\pi k}{n} + i \sin\frac{2\pi k}{n}$. Prove that, for any $1 \le j < n$,

$$\sum_{1 \le k_1 < k_2 < \cdots < k_j \le n} \zeta_{k_1} \cdots \zeta_{k_j} = 0, \qquad \prod_{k=1}^{n} \zeta_k = (-1)^{n+1}$$

and

$$\prod_{k=1}^{n} (1 + \zeta_k) = 1 - (-1)^n, \qquad \prod_{k=1}^{n-1} (1 - \zeta_k) = n.$$