MATH 141, FALL 2016, HW12

DUE IN SECTION, DECEMBER 1

Problem 1. A sequence $\{a_n\}_{n=1}^{\infty}$ is defined recursively in terms of a_1 and a_2 by

$$a_{n+1} = \frac{a_n + a_{n-1}}{2}, \qquad n \geq 2.$$

Prove that for any real a_1 and a_2, $\{a_n\}_{n=1}^{\infty}$ converges, and compute the limit.

Problem 2. A function f satisfies the differential equation

$$xf''(x) + 3x[f'(x)]^2 = 1 - e^{-x}$$

for all real x.

(1) If f has an extremum at a point $c \neq 0$, show that this extremum is a minimum.
(2) If f has an extremum at 0, is it a maximum or a minimum? Justify your conclusion.
(3) If $f(0) = f'(0) = 0$, find the smallest constant A such that $f(x) \leq Ax^2$ for all $x \geq 0$. (Hint: $1 - e^{-x} < x$ for $x > 0$.)

Problem 3. Decide convergence of the following improper integrals.

(1) $\int_{0^+}^{1} \frac{\log x}{\sqrt{x}} dx$
(2) $\int_{0}^{1^-} \frac{\log x}{1-x} dx$
(3) $\int_{0^+}^{1^-} \frac{dx}{\sqrt{x}\log x}$
(4) $\int_{2}^{\infty} \frac{dx}{x(\log x)^3}$.

Problem 4. For a certain real C, the integral

$$\int_1^{\infty} \left(\frac{x}{2x^2 + 2C} - \frac{C}{x+1} \right) dx$$

converges. Determine C and evaluate the integral.

Problem 5. Find the radius of convergence of each of the following power series. Within the radius of convergence, evaluate the series in terms of elementary functions.

(1) $\sum_{n=0}^{\infty} n^3 z^n$
(2) $\sum_{n=0}^{\infty} \frac{2^n}{n^2} z^n$
(3) $\sum_{n=0}^{\infty} \frac{2^n n^2}{n!} z^n$
(4) $\sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n$.

Problem 6. Let $a_n \geq 0$. Prove that the convergence of $\sum_{n=1}^{\infty} a_n$ implies the convergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}.$$

Bonus Problem. Prove that

$$f(x) = \sum_{k \geq 0} \binom{2k}{k} x^k$$

has radius of convergence $\frac{1}{4}$, and, within its radius of convergence,

$$f(x) = \frac{1}{\sqrt{1-4x}}.$$