MATH 141, FALL 2016, HW11

DUE IN SECTION, NOVEMBER 22

Problem 1. (20 points) Justify that the following series converge and have the given value.

(1) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$.
(2) $\sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = 3$.
(3) $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$.
(4) $\sum_{n=1}^{\infty} \frac{2^n+3^n}{6^n} = \frac{3}{2}$.
(5) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$.

Problem 2. Prove that the following sequence converges and find its limit:

$$\sqrt{1}, \sqrt{1+\sqrt{1}}, \sqrt{1+\sqrt{1+\sqrt{1}}}, \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}, \ldots.$$

(Hint: first guess the limit.)

Problem 3. Prove that the sequence defined by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$$

converges. The limit, usually denoted γ, is Euler's constant.

Problem 4. Prove that the sequence

$$\frac{1}{2}, \frac{1}{2+\frac{1}{2}}, \frac{1}{2+\frac{1}{2+\frac{1}{2}}}, \ldots$$

converges to $\sqrt{2}-1$.

Problem 5. Let a_{ij} be the number in the ith row and jth column of the array

$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Prove that

$$\sum_i \left(\sum_j a_{ij} \right) = 0, \qquad \sum_j \left(\sum_i a_{ij} \right) = -2.$$

Problem 6. Prove that $\int_{2\pi}^{\infty} \frac{\sin x}{\sqrt{x}} dx$ converges.

Bonus Problem. If $a_1, a_2, a_3, ...$ is a sequence of positive integers, prove that the sequence

$$s_1 = a_1, s_2 = a_1 + \frac{1}{a_2}, s_3 = a_1 + \frac{1}{a_2 + \frac{1}{a_3}}, s_4 = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4}}}, ...$$

has a real limit x. The sequence $s_1, s_2, ...$ is called the continued fraction expansion of x, and is often denoted $[a_1, a_2, a_3, ...]$. Prove that a positive real number x has a unique finite or infinite continued fraction expansion, and that the expansion is finite if and only if x is rational.

Bonus Problem. A sequence $\{a_n\}$ is said to be *eventually periodic* if there exist positive integers N and p such that, for all $n > N$, $a_{n+p} = a_n$. Prove that if a positive irrational number x has an infinite eventually periodic continued fraction expansion, then x solves a quadratic equation with integer coefficients. The converse is true, but you needn't prove it.