Math 141: Lecture 7

Limits and continuity

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Convexity

Definition

A function f on $[a, b]$ is *convex*, resp. *concave* if, for all $x, y \in [a, b]$ and all $0 \leq \alpha \leq 1$,

$$g(\alpha y + (1-\alpha)x) \leq \alpha g(y) + (1-\alpha)g(x)$$

resp.

$$g(\alpha y + (1-\alpha)x) \geq \alpha g(y) + (1-\alpha)g(x).$$

Properties of indefinite integrals

Theorem

Let $A(x) = \int_a^x f(t)dt$. Then A is convex on every interval on which f is increasing, and concave on every interval on which f is decreasing.

Properties of indefinite integrals

Proof.

Assume f is increasing. Let $x, y \in [a, b]$ with $x < y$. Let $0 < \alpha < 1$ and set $z = \alpha y + (1-\alpha)x$. Write $A(z) = (1-\alpha)A(z) + \alpha A(z)$. It suffices to show

$$(1-\alpha)[A(z) - A(x)] \leq \alpha[A(y) - A(z)],$$

or, writing $z - x = \alpha(y - x)$, $y - z = (1-\alpha)(y-x)$,

$$\frac{A(z) - A(x)}{z - x} \leq \frac{A(y) - A(z)}{y - z}.$$

This follows from the mean property

$$\mathrm{LHS} = \frac{1}{z-x}\int_x^z f(t)dt \leq f(z) \leq \frac{1}{y-z}\int_y^z f(t)dt = \mathrm{RHS}.$$

$\square$

The neighborhood of a point

Definition

Any open interval containing a point p as its midpoint is called a neighborhood of p. For $r > 0$,

$$N(p; r) = \{x : |x - p| < r\}.$$

When considering a point $p \in [a, b]$,

$$N(p; r) = \{x \in [a, b] : |x - p| < r\}.$$

Limits

Definition

We say $f(x)$ has the limit A at p, and write

$$\lim_{x \to p} f(x) = A$$

or, equivalently,

$$f(x) \to A \text{ as } x \to p,$$

if for every neighborhood $N_1(A)$ there is some neighborhood $N_2(p)$ such that $f(x) \in N_1(A)$ whenever $x \in N_2(p)$ and $x \neq p$.

Limits

An equivalent definition:

Definition

- The function $f(x)$ has limit A at p if for every $\epsilon > 0$ there exists $\delta > 0$ such that $0 < |x - p| < \delta$ implies $|f(x) - A| < \epsilon$.
- $f(x)$ has limit A at p *on the right*, resp. *on the left*, written

$$\lim_{x \to p^+} f(x) = A, \qquad \lim_{x \to p^-} f(x) = A$$

  if for every $\epsilon > 0$ there is $\delta > 0$ such that $0 < x - p < \delta$ (resp. $0 < p - x < \delta$) implies $|f(x) - A| < \epsilon$.

Example of limits

- If c is a constant, $\lim_{x\to p} c = c$. For any $\epsilon > 0$, take any $\delta > 0$.
- We have $\lim_{x\to p} x = p$. Given $\epsilon > 0$, choose $\delta = \epsilon$.
- If $p > 0$, then $\lim_{x\to p} x^2 = p^2$. Given $\epsilon > 0$, choose $\delta = \min(\frac{p}{2}, \frac{\epsilon}{2p})$. Then
$$|x^2 - p^2| = |x+p||x-p| < 2p\frac{\epsilon}{2p} = \epsilon.$$

Example of limits

- Let $f(x) = \frac{1}{x^2}$ if $x \neq 0$ and $f(0) = 0$. In this case, f does not have a finite limit at 0. To check this, suppose that $\lim_{x \to 0} f(x) = A$ for some real number A. Choose $\epsilon = 1$. Suppose a $\delta > 0$ has been given. Choose an x with $0 < x < \min(\delta, \frac{1}{1+|A|})$. Then we obtain a contradiction

$$\frac{1}{x^2} \geq (1 + |A|)^2 = 1 + 2|A| + |A|^2$$

so

$$\left| \frac{1}{x^2} - A \right| > 1 + |A|.$$

- Let $f(0) = 0$, $f(x) = 1$ for $x \neq 0$. Then $\lim_{x \to 0} f(x) = 1$. To prove this, given any $\epsilon > 0$, choose any $\delta > 0$.

Definition of continuity

Definition

A function f is said to be continuous at a point p if

1. f is defined at p, and
2. $\lim_{x\to p} f(x) = f(p)$.

A function f on an interval $[a, b]$ is continuous on the left at a (resp. continuous on the right at b) if $\lim_{x\to a^+} f(x) = f(a)$, resp. $\lim_{x\to b^-} f(x) = f(b)$. We often say just f is continuous at a, resp. b.

Definition of continuity

Definition

A function f is said to be continuous on an interval I if it is defined and continuous at each point $p \in I$.

Properties of limits

Theorem

Let f and g be functions such that

$$\lim_{x \to p} f(x) = A, \qquad \lim_{x \to p} g(x) = B.$$

Then we have

1. $\lim_{x \to p}[f(x) + g(x)] = A + B.$
2. $\lim_{x \to p}[f(x) - g(x)] = A - B.$
3. $\lim_{x \to p} f(x) \cdot g(x) = A \cdot B.$
4. $\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{A}{B}$ *if* $B \neq 0.$

Properties of limits

Proof.

$\lim_{x\to p}[f(x)+g(x)] = A+B$:

- Given $\epsilon > 0$, choose $\delta_1, \delta_2 > 0$ such that $|x-p| < \delta_1$ implies $|f(x)-A| < \frac{\epsilon}{2}$ and $|x-p| < \delta_2$ implies $|g(x)-B| < \frac{\epsilon}{2}$.
- Let $\delta = \min(\delta_1, \delta_2)$.
- Then $|x-p| < \delta$ implies

$$|f(x)+g(x)-A-B| \leq |f(x)-A| + |g(x)-B| < \epsilon.$$

The proof for $f-g$ is essentially the same. $\square$

Properties of limits

Proof.

$\lim_{x\to p}[f(x)\cdot g(x)] = A\cdot B$:

- Let $0 < \epsilon < 1$ be given.
- Let $\delta > 0$ be such that $|x-p| < \delta$ implies $|f(x)-A| < \min(\frac{|A|}{2}, \frac{\epsilon}{2|B|})$ and $|g(x)-B| < \min(\frac{|B|}{2}, \frac{\epsilon}{4|A|})$.
- Write
$$f(x)g(x) - AB = f(x)[g(x)-B] + B[f(x)-A].$$
- Thus
$$\begin{aligned}|f(x)g(x) - AB| &\le |f(x)||g(x)-B| + |B||f(x)-A| \\ &< (2|A|)\frac{\epsilon}{4|A|} + |B|\frac{\epsilon}{2|B|} = \epsilon.\end{aligned}$$

$\square$

Properties of limits

Proof.

$\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{A}{B}$:

By combining with the other properties, it suffices to assume $f(x) = 1$ and $B = 1$. Given $\epsilon > 0$, choose $\delta > 0$ such that $|x - p| < \delta$ implies

$$|g(x) - 1| < \min\left(\frac{1}{2}, \frac{\epsilon}{2}\right).$$

For such x,

$$\left|\frac{1}{g(x)} - 1\right| = \frac{|g(x) - 1|}{|g(x)|} < 2\frac{\epsilon}{2} = \epsilon.$$

□

Properties of continuity

Theorem

Let f and g be continuous at a point p. Then $f+g$, $f-g$ and fg are continuous at p. The same is true of $\frac{f}{g}$ if $g(p) \neq 0$.

Proof.

Since $\lim_{x\to p} f(x) = f(p)$ and $\lim_{x\to p} g(x) = g(p)$,
$\lim_{x\to p}(f+g)(x) = f(p) + g(p)$, etc. $\square$

Squeezing principle

Theorem

Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \neq p$ in some neighborhood of p. Suppose also that

$$\lim_{x \to p} f(x) = \lim_{x \to p} h(x) = a.$$

Then $\lim_{x \to p} g(x) = a$.

Proof.

Given $\epsilon > 0$, choose $\delta > 0$ such that $0 < |x - p| < \delta$ implies $|f(x) - a| < \epsilon$ and $|h(x) - a| < \epsilon$. For such x, if $g(x) < a$, then

$$\begin{aligned} a - g(x) \leq a - f(x) < \epsilon, &\qquad \text{if } g(x) < a \\ g(x) - a \leq h(x) - a < \epsilon, &\qquad \text{if } g(x) \geq a. \end{aligned}$$

Thus $|g(x) - a| < \epsilon$. □

Continuity of indefinite integrals

Theorem

Assume f is integrable on $[a, x]$ for every $x \in [a, b]$, and let

$$A(x) = \int_a^x f(t)dt.$$

Then A is continuous at each point of $[a, b]$.

Continuity of indefinite integrals

Proof.

Let $|f| < M$ on $[a, b]$. It follows that on any interval $[c, d] \subset [a, b]$,

$$-M(d-c) \leq \int_c^d f(t)dt \leq M(d-c).$$

To check the continuity at $p \in [a, b]$, given $\epsilon > 0$ choose $\delta = \frac{\epsilon}{M}$. Then for any $x \in [a, b]$, $|x - p| < \delta$ implies

$$|A(p) - A(x)| = \left| \int_x^p f(t)dt \right| < M\delta = \epsilon.$$

$\square$

Examples of continuous functions

- Polynomials. These are obtained by taking sums and products starting from the constant function and $f(x) = x$.
- Rational functions. A rational function $r(x) = \frac{p(x)}{q(x)}$ is the ratio of two polynomials. This is defined and continuous wherever $q(x) \neq 0$.
- The trig functions, $\sin, \cos, \tan, \cot, \sec, \csc$. Both $\sin$ and $\cos$ may be expressed as indefinite integrals. The remaining functions are obtained by taking quotients.

Example of the squeeze principle

By applying the squeeze principle on $[0, \frac{\pi}{2}]$ to

$$0 < \cos x < \frac{\sin x}{x} < 1$$

obtain

$$\lim_{x \to 0^+} \frac{\sin x}{x} = 1.$$

Composition of continuous functions

Theorem

Let f be continuous at p and let g be continuous at $f(p)$. Then $g \circ f$ is continuous at p.

Proof.

Given $\epsilon > 0$, let $\eta > 0$ be such that $|y - f(p)| < \eta$ implies $|g(y) - g(f(p))| < \epsilon$. Let $\delta > 0$ be such that $|x - p| < \delta$ implies $|f(x) - f(p)| < \eta$. Then $|x - p| < \delta$ implies that $|f(x) - f(p)| < \eta$, which implies

$$|g \circ f(x) - g \circ f(p)| < \epsilon.$$

□

The sign of a continuous function

Theorem

Let f be continuous at c and suppose that $f(c) \neq 0$. Then there is a neighborhood of c on which f has the same sign as $f(c)$.

Proof.

Choose $\epsilon = \frac{|f(c)|}{2}$ in the definition of continuity. For the corresponding δ, the claim holds on $N(c, \delta)$. $\square$

Bolzano's theorem

Theorem (Bolzano's theorem)

Let f be continuous on $[a, b]$ and suppose that $f(a)$ and $f(b)$ have opposite signs. Then there is a c, $a < c < b$ such that $f(c) = 0$.

Proof.

Without loss of generality, let $f(a) < 0$ and $f(b) > 0$. Let $S = \{x : f(x) < 0\}$ and set $c = \sup S$. By the previous theorem, if $f(c) \neq 0$ then there is a neighborhood about c on which f has the same sign as c, which contradicts the fact that it is the least upper bound. It follows that $f(c) = 0$.

$\square$

The intermediate value theorem

Theorem (Intermediate value theorem)

Let f be continuous on $[a, b]$. For every value y between $f(a)$ and $f(b)$ there is $c \in [a, b]$ satisfying $f(c) = y$.

Proof.

Let $g(x) = f(x) - y$. Then g changes sign on $[a, b]$, and hence, by Bolzano's theorem $g(c) = 0$ has a solution, which also solves $f(c) = y$. $\square$

The inverse of a monotonic continuous function

Theorem

Let f be strictly increasing and continuous on $[a, b]$, satisfying $f(a) = c$ and $f(b) = d$. Then f^{-1} exists and is strictly increasing and continuous on $[c, d]$.

Proof.

- Since f is strictly increasing it is injective.
- It is surjective by the intermediate value theorem.
- Thus $f^{-1} : [c, d] \to [a, b]$ is well-defined.
- Given $u < v$ find x, y such that $f(x) = u$, $f(y) = v$. Then $x < y$, so f^{-1} is strictly increasing.

$\square$

The inverse of a monotonic continuous function

Proof.

- To prove that f^{-1} is continuous at $y = f(x) \in [c, d]$, given $\epsilon > 0$ solve $f^{-1}(y_1) = \max(x - \frac{\epsilon}{2}, a)$, $f^{-1}(y_2) = \min(x + \frac{\epsilon}{2}, b)$, and, if $y_1 \neq y$ and $y_2 \neq y$,

$$\delta = \min(y - y_1, y_2 - y),$$

  otherwise the only one which is non-zero.
- Then $\delta > 0$ and if $y' \in [c, d]$ satisfies $|y' - y| < \delta$, then $y_1 \leq y' \leq y_2$, which implies

$$|f^{-1}(y') - f^{-1}(y)| \leq \frac{\epsilon}{2}.$$

$\square$

Extreme-value theorem for continuous functions

Theorem (Extreme-value theorem)

Let f be continuous on $[a, b]$. Then f is bounded on $[a, b]$.

Extreme-value theorem for continuous functions

Proof.

Suppose f is unbounded.

1. Given f unbounded on $[a,b]$, it is unbounded on either $[a,(a+b)/2]$ or $[(a+b)/2,b]$ (or both).
2. Form a sequence of intervals $[a_1,b_1]=[a,b]$, $[a_2,b_2]$, $[a_3,b_3]$, ... by letting each successive interval be a half of the previous interval on which f is unbounded.
3. Let $\alpha=\sup\{a_n : n=1,2,3,...\}$. Then $\alpha\in[a,b]$ and f is continuous at α, so there is a neighborhood $N(\alpha,\delta)$ on which $|f(x)|\leq|f(\alpha)|+1$.
4. Since the sequence a_n is increasing, $\alpha-\delta<a_n\leq\alpha$ for all n sufficiently large, whence $\alpha-\delta<b_n<\alpha+\delta$ for all n sufficiently large. This contradicts f unbounded on $[a_n,b_n]$ for all n.

$\square$

Extreme-value theorem for continuous functions

Theorem

Let f be continuous on $[a, b]$. Then f achieves its maximum and minimum on $[a, b]$.

Proof.

Let $M = \sup f$ and let $g = M - f$. Suppose for contradiction that there does not exist a $c \in [a, b]$ with $f(c) = M$. Then $g > 0$ on $[a, b]$, whence $\frac{1}{g}$ is continuous and bounded on $[a, b]$, say by $C > 0$. Then $g \geq \frac{1}{C}$ and $f \leq M - \frac{1}{C}$, a contradiction. $\square$

Uniform continuity

Definition

A function f is said to be *uniformly continuous* on $[a, b]$ if, for all $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Theorem

Let f be continuous on $[a, b]$. Then f is uniformly continuous on $[a, b]$.

Uniform continuity

Let $\epsilon > 0$. Say that a function f on $[a, b]$ 'satisfies the uniform continuity property with parameter ϵ' if there exists $\delta > 0$ such that, if $x, y \in [a, b]$ are such that $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Lemma

Let $\epsilon > 0$. Let f be continuous on $[a, b]$ and let $a < c < b$. If f satisfies the uniform continuity property with parameter ϵ on $[a, c]$ and on $[c, b]$ then it does so on $[a, b]$, also.

Uniform continuity

Proof.

- Let $\delta_1 > 0$ be such that, if $|x - y| < \delta_1$ and both $x, y \in [a, c]$ or both $x, y \in [c, b]$ then $|f(x) - f(y)| < \epsilon$.
- By the continuity of f at c, let $\delta_2 > 0$ be such that if $|x - c| < \delta_2$ then $|f(x) - f(c)| < \frac{\epsilon}{2}$.
- Choose $\delta = \min(\delta_1, \delta_2)$.
- To check the uniform continuity property at ϵ of f on $[a, b]$, it suffices to consider the case $|x - y| < \delta$, $x \in [a, c]$, $y \in [c, b]$.
- In this case, $|c - x| + |y - c| = |y - x| < \delta$, so

$$|f(x) - f(y)| \leq |f(x) - f(c)| + |f(c) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

Uniform continuity

Proof that continuity implies uniform continuity on a closed interval.

Suppose for contradiction that there is an $\epsilon > 0$ for which the uniform continuity property at ϵ does not hold on $[a, b]$.

- If uniform continuity with parameter ϵ does not hold on $[a, b]$, then it does not hold on one of $[a, (a + b)/2]$, $[(a + b)/2, b]$.
- Perform the method of bisection to obtain intervals $[a_1, b_1], [a_2, b_2], \ldots$ on which uniform continuity does not hold with parameter ϵ.
- Let $\alpha = \sup\{a_n\}$. By continuity at α with parameter $\frac{\epsilon}{2}$, there is a neighborhood $N(\alpha, \delta)$ on which $|f(x) - f(\alpha)| < \frac{\epsilon}{2}$.
- By the triangle inequality, for $x, y \in N(\alpha, \delta)$,

$$|f(x) - f(y)| \leq |f(x) - f(\alpha)| + |f(\alpha) - f(y)| < \epsilon.$$

- Since $[a_n, b_n] \subset N(\alpha, \delta)$ for all sufficiently large n, we obtain a contradiction.

$\square$

Integrability of continuous functions

Theorem

Let f be continuous on $[a, b]$. Then f is integrable on $[a, b]$.

Proof.

- By the previous theorem, f is uniformly continuous on $[a, b]$.
- Let $\epsilon > 0$, and choose $n \geq 1$ such that $|x - y| < \frac{b-a}{n}$ implies $f(x) - f(y) < \epsilon$.
- Perform the equipartition of $[a, b]$ into n intervals and form lower and upper step functions s_n, t_n for f on $[a, b]$ by setting s_n to be the minimum of f on each interval, and t_n the max.
- By the uniform continuity, $t_n - s_n < \epsilon$, which shows that $\int_a^b t_n \leq \int_a^b s_n + \epsilon(b - a)$.

It follows that the lower and upper integrals are equal. $\square$

Mean value theorem for integrals

Theorem (Mean value theorem for integrals)

Let f be continuous on $[a, b]$. Then there is $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

Proof.

Let $A = \frac{1}{b-a}\int_a^b f(x)dx$. Let m and M denote the min and max of f on $[a, b]$. Then $m \leq A \leq M$, so that the conclusion follows by the intermediate value theorem. $\square$

Open intervals

The function $f(x) = \frac{1}{x}$ on $(0, 1]$ gives an example of a continuous function which is

- Not bounded
- Not uniformly continuous
- Not integrable

Hence, the fact that we work with closed intervals is necessary in the proofs of the last several theorems.

Further notions of limits

Several further notions of limits exist other than those already discussed. Any interval (M, ∞) is called a 'neighborhood of infinity' while any interval $(-\infty, M)$ is called a 'neighborhood of negative infinity'.

- $\lim_{x\to\infty} f(x) = A$ if, for all $\epsilon > 0$ there exists $M > 0$ such that $x > M$ implies $f(x)$ is defined and $|f(x) - A| < \epsilon$.
- $\lim_{x\to p} f(x) = \infty$ if, for all $N > 0$ there exists $\delta > 0$ such that $0 < |x - p| < \delta$ implies $f(x) > N$.

Further notions of limits

Both of the previous definitions fits the following common generalization.

Definition

Let $p, q \in \mathbb{R} \cup \{\infty, -\infty\}$ (this set is sometimes called the 'extended real numbers'). We write $\lim_{x\to p} f(x) = q$ if, for every neighborhood N_1 of q there is a neighborhood N_2 of p, such that if $x \in N_2 \setminus \{p\}$ then $f(x) \in N_1$.

Further notions of limits

If f is a function defined on $\mathbb{N}$ (also called a sequence), then

- $\lim_{n\to\infty} f(n) = A$ means, for all $\epsilon > 0$ there exists $N > 0$, such that $n > N$ implies $|f(n) - A| < \epsilon$.
- $\lim_{n\to\infty} f(n) = \infty$ means, for all $M > 0$ there exists $N > 0$, such that $n > N$ implies $f(n) > M$.