Math 141: Lecture 5

Area axioms, definition of the integral

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Area of sets

- The main goal of integration theory is to assign area to subsets of $\mathbb{R}^2$.
- We don't have a satisfactory way of assigning area to all subsets. Those subsets to which we can assign an area are called measurable.
- The collection of measureable subsets is written $\mathscr{M}$.

Area axioms

Area a is defined to satisfy the following axioms.

- *Nonnegative property*. For each set S in $\mathscr{M}$, we have $a(S) \geq 0$.
- *Additive property*. If S and T are in $\mathscr{M}$, then $S \cup T$ and $S \cap T$ are in $\mathscr{M}$, and we have

$$a(S \cup T) = a(S) + a(T) - a(S \cap T).$$

- *Difference property*. If S and T are in $\mathscr{M}$ with $S \subset T$, then $T \setminus S$ is in $\mathscr{M}$, and we have $a(T \setminus S) = a(T) - a(S)$.

Area axioms

- *Invariance under congruence*. Say two sets S and T are congruent if there is a bijection $S \to T$ which preserves lengths. If $S \in \mathscr{M}$ and S and T are congruent, then $T \in \mathscr{M}$ and $a(S) = a(T)$.
- *Choice of scale*. Every rectangle R is in $\mathscr{M}$. If the edges of R have lengths h and k, then $a(R) = hk$.
- *Exhaustion property*. A step region is the union of several adjacent rectangles. Let Q be a set enclosed between two step regions S, T, so that $S \subset Q \subset T$. If there is one and only one number c which satisfies

$$a(S) \leq c \leq a(T)$$

for all pairs of step regions S, T enclosing Q, then $a(Q) = c$.

Partitions

- Let $[a, b]$ be a closed interval.
- A collection of points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

  is called a *partition* of $[a, b]$.
- The partition is indicated $P = \langle x_0, x_1, ..., x_n \rangle$.
- The partition P determines open *subintervals* (x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n).
- The *common refinement* of two partitions P_1, P_2 is $P = P_1 \cup P_2$ (points taken in order).

Step functions

A function s whose domain is the closed interval $[a, b]$ is called a *step function* if there is a partition $P = \langle x_0, x_1, ..., x_n \rangle$ of $[a, b]$ such that s is constant on each open subinterval. In particular, for each $1 \le k \le n$, there is s_k such that

$$s(x) = s_k, \qquad \text{if } x_{k-1} < x < x_k.$$

We say that s is subordinate to P.

Step functions

Let P_1, P_2 be two partitions of $[a, b]$ such that s is subordinate to both P_1 and P_2. Then s is subordinate to $P_1 \cap P_2$. It follows that there is a partition P of minimal cardinality such that s is subordinate to P. Any partition P' of $[a, b]$ to which s is subordinate can be obtained by adding one or several points to P.

Sums and products of step functions

- Let s and t be step functions on $[a, b]$. Let P_1 and P_2 be partitions of $[a, b]$ such that s is constant on the open subintervals of P_1, and t on those of P_2.
- Let P be the common refinement of P_1, P_2.
- Then s and t are both constant on the open subintervals of P. In particular, $s + t$ and st are step functions which are constant on the open subintervals of P.

The integral of a step function

Let s be a step function on $[a, b]$, subordinate to the partition $P = \langle x_0, x_1, ..., x_n \rangle$. Suppose that

$$s(x) = s_k \qquad \text{if } x_{k-1} < x < x_k.$$

Definition

The integral of s from a to b, denoted $\int_a^b s(x)dx$, is defined by the following formula:

$$\int_a^b s(x)dx = \sum_{k=1}^{n} s_k \cdot (x_k - x_{k-1}).$$

The integral of a step function

The integral of a step function s is independent of the partition chosen.

- Suppose that s is subordinate to a partition P and add a single additional point t between x_{k-1} and x_k.
- The interval (x_{k-1}, x_k) is split into the two intervals (x_{k-1}, t) and (t, x_k).
- In the new sum, $s_k(x_k - x_{k-1})$ is replaced by $s_k(t - x_{k-1}) + s_k(x_k - t)$, which leaves the sum unchanged.
- Any partition to which s is subordinate can be obtained by adding one or more points to the minimal partition, so each obtains the same integral.

Properties of the integral of a step function

Let s and t be step functions on $[a, b]$.

Theorem (Additive property)

$$\int_a^b [s(x) + t(x)]dx = \int_a^b s(x)dx + \int_a^b t(x)dx.$$

Proof.

Choose a partition $P = \langle x_0, x_1, ..., x_n \rangle$ to which both s, t are subordinate. Then

$$\int_a^b [s(x) + t(x)]dx = \sum_{k=1}^{n} (s_k + t_k)(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} s_k(x_k - x_{k-1}) + \sum_{k=1}^{n} t_k(x_k - x_{k-1}) = \int_a^b s(x)dx + \int_a^b t(x)dx.$$

$\square$

Properties of the integral of a step function

Theorem (Homogeneous property)

For every real number c, we have

$$\int_a^b c \cdot s(x)dx = c\int_a^b s(x)dx.$$

Proof.

$$\begin{aligned}\int_a^b c \cdot s(x)dx &= \sum_{k=1}^n cs_k(x_k - x_{k-1}) \\ &= c\sum_{k=1}^n s_k(x_k - x_{k-1}) = c\int_a^b s(x)dx.\end{aligned}$$

$\square$

Properties of the integral of a step function

Theorem (Linearity property)

For every real c_1 and c_2, we have

$$\int_a^b [c_1 s(x) + c_2 t(x)]dx = c_1 \int_a^b s(x)dx + c_2 \int_a^b t(x)dx.$$

Proof.

Combine the previous two theorems. $\square$

Properties of the integral of a step function

Theorem (Comparison theorem)

If $s(x) < t(x)$ *for every* $x \in [a, b]$*, then*

$$\int_a^b s(x)dx < \int_a^b t(x)dx.$$

Proof.

Write

$$\int_a^b t(x)dx - \int_a^b s(x)dx = \sum_{k=1}^{n} (t_k - s_k)(x_k - x_{k-1}).$$

Being a sum of non-negative terms, the difference of integrals is non-negative. $\square$

Properties of the integral of a step function

Theorem (Additivity with respect to the interval of integration)

Let $a < c < b$. Then

$$\int_a^c s(x)dx + \int_c^b s(x)dx = \int_a^b s(x)dx.$$

Proof.

Let $P = \langle x_0, x_1, ..., x_n \rangle$ be a partition of $[a, b]$ which includes $x_m = c$ for some $0 < m < n$. Suppose $s(x)$ is subordinate to P. Then

$$\begin{aligned}\int_a^c s(x)dx + \int_c^b s(x)dx &= \sum_{k=1}^{m} s_k(x_k - x_{k-1}) + \sum_{k=m+1}^{n} s_k(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} s_k(x_k - x_{k-1}) = \int_a^b s(x)dx.\end{aligned}$$

$\square$

Properties of the integral of a step function

Theorem (Invariance under translation)

For every real c,

$$\int_a^b s(x)dx = \int_{a+c}^{b+c} s(x-c)dx.$$

Proof.

Let $s(x)$ be subordinate to the partition $P = \langle x_0, x_1, ..., x_n \rangle$. Then $s(x-c)$ is subordinate to the partition $P + c = \langle x_0 + c, x_1 + c, ..., x_n + c \rangle$. Thus

$$\begin{aligned}\int_a^b s(x)dx &= \sum_{k=1}^n s_k(x_k - x_{k-1}) \\ &= \sum_{k=1}^n s_k(x_k + c - (x_{k-1} + c)) = \int_{a+c}^{b+c} s(x-c)dx.\end{aligned}$$

$\square$

Properties of the integral of a step function

Define $\int_b^a s(x)dx = -\int_a^b s(x)dx$, and $\int_a^a s(x)dx = 0$.

Theorem (Expansion or contraction of the interval of integration)

For every $c \neq 0$,

$$\int_{ca}^{cb} s\left(\frac{x}{c}\right) dx = c \int_a^b s(x)dx.$$

Proof.

First suppose $c > 0$. Let s be subordinate to $P = \langle x_0, x_1, ..., x_n \rangle$. Then $s\left(\frac{x}{c}\right)$ is subordinate to $cP = \langle cx_0, cx_1, ..., cx_n \rangle$. Hence

$$\int_{ca}^{cb} s\left(\frac{x}{c}\right) dx = \sum_{k=1}^{n} s_k(cx_k - cx_{k-1}) = c \int_a^b s(x)dx.$$

□

Properties of the integral of a step function

Proof.

To prove the remainder of the claim, note that $s(-x)$ is subordinate to $-P = \langle -x_n, -x_{n-1}, ..., -x_0 \rangle$, and hence

$$\begin{aligned}\int_{-b}^{-a} s(-x)dx &= \sum_{k=1}^{n} s_{n+1-k}(-x_{n-k} - (-x_{n-k+1})) \\ &= \sum_{k=1}^{n} s_k(x_k - x_{k-1}) = \int_a^b s(x)dx\end{aligned}$$

by making the substitution $k = n - k + 1$. □

The last equality is called the reflection principle.

The integral of a bounded function

- Say f is bounded on $[a, b]$ if there exists an $M > 0$ such that, for all $x \in [a, b]$, $|f(x)| \leq M$.
- Let S denote the set of step functions $s \leq f$, and let T denote the set of step functions $t \geq f$. Both sets are non-empty, since $s(x) = -M$ is in S, and $t(x) = M$ is in T.
- Define the lower and upper integrals of f to be

$$\underline{I}(f) = \sup\left\{\int_a^b s(x)dx : s \in S\right\}, \quad \overline{I}(f) = \inf\left\{\int_a^b t(x)dx : t \in T\right\}.$$

Note $\underline{I}(f) \leq \overline{I}(f)$.

The integral of a bounded function

Definition

A bounded function f on $[a, b]$ is integrable if $\underline{I}(f) = \overline{I}(f)$. In this case, define

$$\int_a^b f(x)dx = \underline{I}(f).$$

Define, also,

$$\int_b^a f(x)dx = -\int_a^b f(x)dx, \qquad \int_a^a f(x)dx = 0.$$

The graph of an integrable function

Theorem

Let f be a non-negative integrable function on $[a, b]$, and let

$$Q = \{(x, y) : a \leq x \leq b, 0 \leq y \leq f(x)\}$$

denote the ordinate set of f. Then Q is measurable, and its area is equal to

$$\int_a^b f(x)dx.$$

The graph of an integrable function

Proof.

- Let s, t be step functions with $s \leq f \leq t$. The ordinate sets of these step functions define step regions S and T with $S \subset Q \subset T$.
- The area of S is the integral of s and the area of T is the integral of t.
- It follows that the only real number which lies between the area of S and the area of T for all step regions $S \subset Q \subset T$ is $\int_a^b f(x)dx$ (exhaustion).

$\square$

The graph of an integrable function

Theorem

Let f be a nonnegative function, integrable on an interval $[a, b]$. Then the graph of f, that is, the set

$$\{(x, y) : a \leq x \leq b, y = f(x)\}$$

is measurable and has area equal to 0.

Proof.

Let

$$Q' = \{(x, y) : a \leq x \leq b, 0 \leq y < f(x)\}.$$

Modify the rectangles used in the step regions S from the previous theorem to exclude their boundary, without changing the measurability or area. The argument now shows that Q' is measurable with area equal to Q, and hence the graph of f, which is $Q \setminus Q'$, is measurable, with area 0. $\square$

Monotonic functions

- A function f is monotone increasing (decreasing) on $[a, b]$ if $x < y$ implies $f(x) \leq f(y)$ $(f(x) \geq f(y))$.
- A function f is strictly increasing (decreasing) on $[a, b]$ if $x < y$ implies $f(x) < f(y)$ $(f(x) > f(y))$.
- A function is (strictly) monotonic on $[a, b]$ if it is either (strictly) monotone increasing or (strictly) monotone decreasing.

Examples of Monotonic functions

- If p is a positive integer, it follows by induction that
$$x^p < y^p \qquad \text{if } 0 \leq x < y.$$
- Let $f(x) = \sqrt{x}$ for $x \geq 0$. This function is strictly increasing, since for $y > x$
$$\sqrt{y} - \sqrt{x} = \frac{y - x}{\sqrt{y} + \sqrt{x}} > 0.$$
- In fact, if $n \geq 1$ is any positive integer and $0 \leq x < y$
$$y^{\frac{1}{n}} - x^{\frac{1}{n}} = \frac{y - x}{y^{\frac{n-1}{n}} + y^{\frac{n-2}{n}} x^{\frac{1}{n}} + \ldots + x^{\frac{n-1}{n}}} > 0$$
so $x^{\frac{1}{n}}$ is strictly increasing. Thus x^r is strictly increasing in $x \geq 0$ for any positive rational r.

Monotonic functions are integrable

Theorem

Let f be monotonic on $[a, b]$. Then f is integrable on $[a, b]$.

Proof.

Assume that f is increasing. Let P_n be the partition of $[a, b]$ which divides the interval into n equal intervals. Thus $P = \langle x_0, x_1, ..., x_n \rangle$, and $x_k = a + \frac{k}{n}(b - a)$. Define two step functions $s_n \leq f \leq t_n$ for each k by

$$s_n(x) = f(x_{k-1}), \quad t_n(x) = f(x_k), \qquad x_{k-1} \leq x < x_k,$$

and $s_n(b) = t_n(b) = f(b)$.

□

Monotonic functions are integrable

Proof.

Then

$$\int_a^b t_n(x)dx - \int_a^b s_n(x)dx = \frac{1}{n}\sum_{k=1}^n f(x_k) - \frac{1}{n}\sum_{k=1}^n f(x_{k-1}) = \frac{f(b)-f(a)}{n}.$$

Note

$$\int_a^b s_n(x)dx \le \underline{I}(f) \le \overline{I}(f) \le \int_a^b t_n(x)dx$$

and thus, for each $n = 1, 2, 3, \ldots$

$$0 \le \overline{I}(f) - \underline{I}(f) \le \frac{f(b)-f(a)}{n}.$$

Thus, $\overline{I}(f) = \underline{I}(f)$. □

The integral of a power function

Theorem

Let $p \geq 1$ be an integer. Then $\int_0^b x^p dx = \frac{b^{p+1}}{p+1}$.

Proof.

One has, for $n = 1, 2, 3, ...$

$$\sum_{j=0}^{n-1} j^p < \frac{n^{p+1}}{p+1} < \sum_{j=1}^{n} j^p$$

see HW4. In the context of the proof of the previous theorem,

$$\int_0^b s_n(x)dx = \frac{b}{n}\sum_{k=0}^{n-1}\left(\frac{kb}{n}\right)^p < \frac{b^{p+1}}{p+1} < \frac{b}{n}\sum_{k=1}^{n}\left(\frac{kb}{n}\right)^p = \int_0^b t_n(x)dx.$$

Since this holds for each n, the evaluation follows. □

Properties of the integral

Theorem (Linearity with respect to integrand)

If f and g are integrable on $[a, b]$, then for every pair of constants c_1, c_2, $c_1 f + c_2 g$ is integrable on $[a, b]$. Furthermore,

$$\int_a^b [c_1 f(x) + c_2 g(x)]dx = c_1 \int_a^b f(x)dx + c_2 \int_a^b g(x)dx.$$

Properties of the integral

Theorem (Additivity with respect to the interval of integration)

If two of the following three integrals exist, the third also exists, and we have

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx.$$

Properties of the integral

Theorem (Invariance under translation)

If f is integrable on $[a, b]$, then for every real c we have

$$\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x-c)dx.$$

Properties of the integral

Theorem (Expansion or contraction of the interval of integration)

If f is integrable on $[a, b]$, then for every real $k \neq 0$ we have

$$\int_a^b f(x)dx = \frac{1}{k}\int_{ka}^{kb} f\left(\frac{x}{k}\right) dx.$$

Properties of the integral

Theorem (Comparison theorem)

If both f and g are integrable on $[a, b]$ and if $g(x) \leq f(x)$ for every x in $[a, b]$, then we have

$$\int_a^b g(x)dx \leq \int_a^b f(x)dx.$$