Math 141: Lecture 23

Fourier series and convolution

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Fourier series

Definition

Let f be integrable on $[0,1]$. The *Fourier coefficients* of f are defined by

$$\hat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}dx.$$

The Fourier series of f is the series

$$f(x) \sim \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}.$$

No equality is asserted by the notation $\sim$. We say that the Fourier series converges to f at the point x if

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}(n)e^{2\pi inx}.$$

Fourier series

- The integral formula implies the following *orthogonality relation*

$$\int_0^1 e^{2\pi imx}\overline{e^{2\pi inx}}dx = \int_0^1 e^{2\pi i(m-n)x}dx = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}.$$

- A *trigonometric polynomial* is a finite sum $P(x) = \sum_{n\in S} c_n e^{2\pi inx}$ where S is a finite set of frequencies.
- Calculate, using orthogonality,

$$\int_0^1 |P(x)|^2 dx = \int_0^1 \sum_{n_1,n_2\in S} c_{n_1}\overline{c_{n_2}}e^{2\pi i(n_1-n_2)x}dx = \sum_{n\in S}|c_n|^2.$$

Bessel's inequality

Theorem

Let $S \subset \mathbb{Z}$ be a finite set of frequencies. For any constants $\{c_n\}_{n \in S}$,

$$\int_0^1 \left| f(x) - \sum_{n \in S} \hat{f}(n) e^{2\pi i n x} \right|^2 dx \leq \int_0^1 \left| f - \sum_{n \in S} c_n e^{2\pi i n x} \right|^2 dx.$$

Bessel's inequality

Proof.

Expand the square using
$|A-B|^2 = (A-B)(\overline{A}-\overline{B}) = |A|^2 - A\overline{B} - \overline{A}B + |B|^2$ to write

$$\int_0^1 \left| f - \sum_{n\in S} c_n e^{2\pi i n x}\right|^2 dx = \int_0^1 |f(x)|^2\, dx - \int_0^1 f(x) \sum_{n\in S} \overline{c_n} e^{-2\pi i n x} dx$$

$$- \int_0^1 \overline{f}(x) \sum_{n\in S} c_n e^{2\pi i n x} dx + \int_0^1 \left| \sum_{n\in S} c_n e^{2\pi i n x}\right|^2 dx.$$

Using the orthogonality relation, this becomes

$$\int_0^1 |f(x)|^2\, dx - \sum_{n\in S} \hat{f}(n)\overline{c_n} - \sum_{n\in S} \overline{\hat{f}(n)} c_n + \sum_{n\in S} |c_n|^2.$$

$\square$

Bessel's inequality

Proof.

Rewrite

$$
\begin{aligned}
&\int_0^1 \left| f - \sum_{n \in S} c_n e^{2\pi i n x} \right|^2 dx \\
&\int_0^1 |f(x)|^2 \, dx - \sum_{n \in S} \hat{f}(n)\overline{c_n} - \sum_{n \in S} \overline{\hat{f}(n)} c_n + \sum_{n \in S} |c_n|^2 \\
&= \int_0^1 |f(x)|^2 \, dx - \sum_{n \in S} \left| \hat{f}(n) \right|^2 + \sum_{n \in S} \left| \hat{f}(n) - c_n \right|^2 .
\end{aligned}
$$

This is minimized if $c_n = \hat{f}(n)$ for each $n \in S$, which completes the proof. $\square$

Bessel's inequality

Theorem (Bessel's inequality)

Let f be Riemann integrable on $\mathbb{R}/\mathbb{Z}$ and let $S \subset \mathbb{Z}$ be a possibly infinite set. We have

$$\sum_{n \in S} \left|\hat{f}(n)\right|^2 \leq \int_0^1 |f(x)|^2 dx.$$

Proof.

By the previous proof, for any N,

$$\int_0^1 |f(x)|^2 dx - \sum_{n \in S, |n| \leq N} \left|\hat{f}(n)\right|^2$$

$$= \int_0^1 \left| f(x) - \sum_{n \in S, |n| \leq N} \hat{f}(n) e^{2\pi i n x} \right|^2 dx > 0$$

Taking $N \to \infty$ completes the proof. □

Riemann-Lebesgue Lemma

Lemma (Riemann-Lebesgue Lemma)

Let f be integrable on $\mathbb{R}/\mathbb{Z}$. As $|n| \to \infty$, $\left|\hat{f}(n)\right| \to 0$.

Proof.

This follows, since the sum $\sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2$ converges. □

Convergence of Fourier series

Theorem

Let f be a function on $\mathbb{R}/\mathbb{Z}$ which satisfies for some x, there are constants $\delta > 0$ and $M < \infty$ such that

$$|f(x+t) - f(x)| \leq M|t|$$

for all $t \in (-\delta, \delta)$. Then

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x}.$$

Convergence of Fourier series

The proof uses Dirichlet's kernel $D_N(x) = \sum_{n=-N}^{N} e^{2\pi i n x}$. Recall that this satisfies

- $\int_0^1 D_N(x) dx = 1$
- $D_N(x) = \frac{\sin 2\pi\left(N+\frac{1}{2}\right)x}{\sin \pi x}$

Convergence of Fourier series

Proof.

Since $\int_0^1 D_N(x)dx = 1$, $f(x) = \int_0^1 f(x)D_N(t)dt$. Calculate

$$\sum_{n=-N}^{N} \hat{f}(n)e^{2\pi inx} = \int_0^1 f(t)e^{-2\pi in(t-x)}dt = \int_0^1 f(t)D_N(x-t)dt.$$

By the change of variables $u = x - t$, and the 1-periodicity of f and D_N, this is

$$\int_{x-1}^{x} f(x-t)D_N(t)dt = \int_0^1 f(x-t)D_N(t)dt.$$

Hence

$$f(x) - \sum_{n=-N}^{N} \hat{f}(n)e^{2\pi inx} = \int_0^1 (f(x) - f(x-t))D_N(t)dt.$$

□

Convergence of Fourier series

Proof.

Recall

$$f(x) - \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x} = \int_0^1 (f(x) - f(x-t)) D_N(t) dt.$$

Define

$$g(t) = \begin{cases} \frac{f(x)-f(x-t)}{\sin \pi t} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

which is a bounded, integrable function of t. Write

$$\sin 2\pi \left(N + \frac{1}{2}\right) t = \sin \pi t \cos 2\pi N t + \cos \pi t \sin 2\pi N t.$$

□

Convergence of Fourier series

Recall

$$f(x) - \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x} = \int_0^1 g(t) \left(\sin \pi t \cos 2\pi N t + \cos \pi t \sin 2\pi N t\right) dt.$$

Since $g(t) \sin \pi t$ and $g(t) \cos \pi t$ are integrable, by the Riemann Lebesgue lemma, the integral on the right tends to 0 as $N \to \infty$.

Convolution and Fourier series

Let f and g be two integrable functions on $\mathbb{R}/\mathbb{Z}$. Their convolution is defined by

$$f * g(x) = \int_0^1 f(t)g(x-t)dt = \int_0^1 f(x-t)g(t)dt.$$

The equality between the two integrals is obtained by substituting $u = x - t$ as before.

$f * g$ is periodic with period 1, since $g(x+1-t) = g(x-t)$.

Convolution and Fourier series

Theorem

Let f and g be Riemann integrable on $\mathbb{R}/\mathbb{Z}$. Then $f * g$ is continuous.

Convolution and Fourier series

Proof.

Let $|g| \leq M$. Given $\epsilon > 0$, recall from Homework #6, Problem 2 that there exists a continuous function f_1 such that $\int_0^1 |f(x) - f_1(x)|dx < \frac{\epsilon}{3M}$. Observe

$$
\begin{aligned}
|f * g(x) - f_1 * g(x)| &= \left| \int_0^1 (f(t) - f_1(t))g(x-t)dt \right| \\
&\leq M \int_0^1 |f(t) - f_1(t)|dt < \frac{\epsilon}{3}.
\end{aligned}
$$

□

Convolution and Fourier series

Proof.

Since f_1 is absolutely continuous, there is $\delta > 0$ such that, for $x \in \mathbb{R}$, if $|x-y| < \delta$ then $|f_1(x) - f_1(y)| < \frac{\epsilon}{3M}$. Thus for $|x-y| < \delta$,

$$\begin{aligned} |f_1 * g(x) - f_1 * g(y)| &= \left| \int_0^1 g(t)(f_1(x-t) - f_1(y-t))dt \right| \\ &\leq M \int_0^1 |f_1(x-t) - f_1(y-t)| dt < \frac{\epsilon}{3}. \end{aligned}$$

Combining these estimates,

$$\begin{aligned} &|f * g(y) - f * g(x)| = \\ &|(f * g(y) - f_1 * g(y)) + (f_1 * g(y) - f_1 * g(x)) + (f_1 * g(x) - f * g(x))| \\ &\leq |f * g(y) - f_1 * g(y)| + |f_1 * g(y) - f_1 * g(x)| + |f_1 * g(x) - f * g(x)| \\ &< \epsilon. \end{aligned}$$

□

Convolution and Fourier series

Theorem

Let f be Riemann integrable and let g be a trigonometric polynomial, that is, $g(x) = \sum_{n\in S} c_n e^{2\pi inx}$ for some finite set of frequencies S. Then

$$f * g(x) = \sum_{n\in S} \hat{f}(n) c_n e^{2\pi inx}.$$

Proof.

Calculate

$$f * g(x) = \int_0^1 f(t) \sum_{n\in S} c_n e^{2\pi in(x-t)} dt$$

$$= \sum_{n\in S} c_n e^{2\pi inx} \int_0^1 f(t) e^{-2\pi int} dt = \sum_{n\in S} \hat{f}(n) c_n e^{2\pi inx}.$$

□

Fejér's kernel

Definition

Let $N \geq 1$. The function

$$K_N(x) = \frac{D_N(x)^2}{2N+1} = \frac{1}{2N+1}\left(\frac{\sin 2\pi(N+\frac{1}{2})x}{\sin \pi x}\right)^2$$

is called *Fejér's kernel*. It has Fourier coefficients

$$\hat{K}_N(n) = \begin{cases} \frac{2N+1-|n|}{2N+1} & |n| \leq 2N \\ 0 & \text{otherwise} \end{cases}.$$

Fejér's kernel

To check the claim regarding the Fourier coefficients, expand the square:

$$\begin{aligned} D_N(x)^2 &= \left(\sum_{n=-N}^{N} e^{2\pi i n x}\right)^2 \\ &= \sum_{n_1, n_2=-N}^{N} e^{2\pi i (n_1+n_2)x} \\ &= \sum_{n=-2N}^{2N} (2N+1-|n|)e^{2\pi i n x}. \end{aligned}$$

Fejér's kernel

Theorem

Fejér's kernel satisfies the following properties.

1. $K_N(x) \geq 0$
2. $\int_0^1 K_N(x)dx = 1$
3. *For each fixed* $\delta > 0$, $\lim_{N\to\infty} \int_\delta^{1-\delta} K_N(x)dx = 0$.

These properties make $K_N(x)$ a 'summability kernel'.

Fejér's kernel

Proof.

The first property holds since K_N is proportional to D_N^2. The second property is the Fourier coefficient of K_N at 0. To prove the third, use

$$K_N(x) = \frac{1}{2N+1}\left(\frac{\sin 2\pi(N+\frac{1}{2})x}{\sin \pi x}\right)^2 \leq \frac{1}{2N+1}\left(\frac{1}{\sin \pi x}\right)^2.$$

Thus

$$\int_\delta^{1-\delta} K_N(x)dx \leq \frac{1}{2N+1}\int_\delta^{1-\delta} \frac{1}{(\sin \pi x)^2}dx$$

tends to 0 as $N \to \infty$. $\square$

Convergence of Cesàro means

Theorem

Let f be continuous on $\mathbb{R}/\mathbb{Z}$. Then $f * K_N$ converges to f uniformly as $N \to \infty$.

Convergence of Cesàro means

- The trigonometric polynomial $f * K_N$ is given by

$$f * K_N(x) = \sum_{n=-2N}^{2N} \frac{2N+1-|n|}{2N+1} \hat{f}(n) e^{2\pi i n x}$$

  and is a Cesàro mean of the Fourier series for f.
- This theorem demonstrates that the trigonometric polynomials are dense in the space of continuous functions on $\mathbb{R}/\mathbb{Z}$.

Convergence of Cesàro means

Proof.

Since f is uniformly continuous, given $\epsilon > 0$ choose $\delta > 0$ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{2}$. Let f be bounded by M, and choose N sufficiently large such that

$$\int_{\delta}^{1-\delta} K_N(x)dx < \frac{\epsilon}{4M}.$$

Then, using that $\int_0^1 K_N(x)dx = 1$,

$$|f(x) - f * K_N(x)| = \left|\int_0^1 (f(x) - f(x-t))K_N(t)dt\right|$$
$$\leq \int_0^1 |f(x) - f(x-t)|K_N(t)dt.$$

□

Convergence of Cesàro means

Proof.

Recall

$$|f(x) - f * K_N(x)| \leq \int_0^1 |f(x) - f(x-t)| K_N(t) dt$$
$$= \int_{-\delta}^{\delta} |f(x) - f(x-t)| K_N(t) dt + \int_{\delta}^{1-\delta} |f(x) - f(x-t)| K_N(t) dt.$$

Note that we've used the one-periodicity to replace the integral of integration with $[-\delta, 1-\delta]$. In the first integral, bound $|f(x) - f(x-t)| < \frac{\epsilon}{2}$ to estimate

$$\int_{-\delta}^{\delta} |f(x) - f(x-t)| K_N(t) dt < \frac{\epsilon}{2} \int_0^1 K_N(t) dt = \frac{\epsilon}{2}.$$

□

Convergence of Cesàro means

Proof.

In the integral from δ to $1-\delta$, bound $|f(x) - f(x-t)| \leq 2M$ to estimate

$$\int_\delta^{1-\delta} |f(x) - f(x-t)| K_N(t) dt \leq 2M \int_\delta^{1-\delta} K_N(t) dt < 2M \frac{\epsilon}{4M} = \frac{\epsilon}{2}.$$

Combining these estimates,

$$\int_0^1 |f(x) - f(x-t)| K_N(t) dt < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

completing the proof. $\square$

Convergence in L^1

Theorem

Let f be integrable on $[0, 1]$. Then for any $N \geq 1$,

$$\int_0^1 |f * K_N(x)| dx \leq \int_0^1 |f(x)| dx.$$

Convergence in L^1

Proof.

By positivity of K_N,

$$\int_0^1 |f * K_N(x)|dx = \int_0^1 \left| \int_0^1 f(t)K_N(x-t) \right| dtdx$$
$$\leq \int_0^1 \int_0^1 |f(t)|K_N(x-t)dtdx$$
$$= \sum_{n=-2N}^{2N} \frac{2N+1-|n|}{2N+1} \int_0^1 \int_0^1 |f(t)|e^{2\pi in(x-t)}dtdx$$
$$= \sum_{n=-2N}^{2N} \frac{2N+1-|n|}{2N+1} \int_0^1 e^{2\pi inx} \int_0^1 |f(t)|e^{-2\pi int}dtdx$$

The inner integral over t is a constant which depends on n but not x. Treating this as fixed we may integrate in x to eliminate all but $n = 0$, which leaves $\int_0^1 |f * K_N(x)|dx \leq \int_0^1 |f(t)|dt$. $\square$

Convergence in L^1

Theorem

Let f be integrable on $[0,1]$. Then

$$\lim_{N\to\infty}\int_0^1 |f(x) - f * K_N(x)|dx = 0.$$

Convergence in L^1

Proof.

Given $\epsilon > 0$, choose continuous f_1 such that $\int_0^1 |f(x) - f_1(x)|dx < \frac{\epsilon}{3}$. Choose N sufficiently large so that $|f_1(x) - f_1 * K_N(x)| < \frac{\epsilon}{3}$, uniformly in x. Then

$$\begin{aligned}
&\int_0^1 |f(x) - f * K_N(x)|dx \\
&= \int_0^1 |(f - f_1)(x) + (f_1 - f_1 * K_N)(x) + ((f - f_1) * K_N)(x)|dx \\
&\leq \int_0^1 |(f - f_1)(x)|dx + \int_0^1 |(f_1 - f_1 * K_N)(x)|dx \\
&\qquad + \int_0^1 |f - f_1| * K_N(x)dx \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

□

Parseval's theorem

Theorem (Parseval's theorem)

Let f and g be Riemann integrable on $\mathbb{R}/\mathbb{Z}$. Then

$$\int_0^1 |f(x)|^2 dx = \sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2$$

and

$$\int_0^1 f(x)\overline{g}(x) dx = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}.$$

Parseval's theorem

Proof.

Observe

$$\hat{\overline{g}}(n) = \int_0^1 \overline{g}(x) e^{-2\pi i n x} dx = \overline{\int_0^1 g(x) e^{2\pi i n x} dx} = \overline{\hat{g}(-n)}.$$

Calculate

$$\begin{aligned}\int_0^1 |f(x)|^2 dx &= \lim_{N\to\infty} \int_0^1 f(x)(\overline{f} * K_N(x))dx \\ &= \lim_{N\to\infty} \int_0^1 f(x) \sum_{n=-2N}^{2N} \frac{2N+1-|n|}{2N+1} \overline{\hat{f}(-n)} e^{2\pi i n x} dx \\ &= \lim_{N\to\infty} \sum_{n=-2N}^{2N} \frac{2N+1-|n|}{2N+1} \left|\hat{f}(-n)\right|^2.\end{aligned}$$

□

Parseval's theorem

Proof.

To check that

$$\lim_{N\to\infty} \sum_{n=-2N}^{2N} \frac{2N+1-|n|}{2N+1} \left|\hat{f}(-n)\right|^2 = \sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2,$$

observe that for each fixed n, $\frac{2N+1-|n|}{2N+1}\left|\hat{f}(n)\right|^2$ increases to $\left|\hat{f}(n)\right|^2$. Thus, as a function of N the left hand side is increasing and bounded above, so converges to a limit, which is at most the right hand side. For each M the limit is bounded below by $\sum_{n=-M}^{M}\left|\hat{f}(n)\right|^2$, which tends to $\sum_{n\in\mathbb{Z}}\left|\hat{f}(n)\right|^2$ as $M\to\infty$. $\square$

Parseval's theorem

Proof.

For the second statement, calculate in the same way

$$\int_0^1 f(x)\overline{g(x)}dx = \lim_{N\to\infty} \sum_{n=-2N}^{2N} \frac{2N+1-|n|}{2N+1} \hat{f}(-n)\overline{\hat{g}(-n)}.$$

Argue as before, using Cauchy-Schwarz to bound the tail

$$\left| \sum_{M<|n|\leq 2N} \left(\frac{2N+1-|n|}{2N+1}\right) \hat{f}(n)\overline{\hat{g}(n)} \right|^2$$

$$\leq \left(\sum_{|n|>M} \left|\hat{f}(n)\right|^2 \right) \left(\sum_{|n|>M} |\hat{g}(n)|^2 \right).$$

$\square$

Determination of $\zeta(2)$

Theorem

$$\zeta(2) = \frac{\pi^2}{6}.$$

Determination of $\zeta(2)$

Proof.

Recall from last lecture that the square signal

$$s(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ -1 & \frac{1}{2} \leq x < 1 \end{cases}$$

has Fourier coefficients

$$\hat{s}(n) = \begin{cases} 0 & n \text{ even} \\ \frac{-2i}{\pi n} & n \text{ odd} \end{cases}.$$

By Parseval,

$$1 = \int_0^1 |s(x)|^2 dx = \sum_{n \in \mathbb{Z}} |\hat{s}(n)|^2 = \frac{8}{\pi^2} \sum_{n>0, \text{ odd}} \frac{1}{n^2}.$$

□

Determination of $\zeta(2)$

Proof.

Thus

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$$
$$= \frac{1}{1-2^{-2}} \sum_{n>0,\ \text{odd}} \frac{1}{n^2} = \frac{4}{3}\frac{\pi^2}{8} = \frac{\pi^2}{6}.$$

□

Algebraic properties of convolution

Theorem

Let f, g, h be integrable functions. Then

1. $f * g = g * f$
2. $f * (g * h) = (f * g) * h$
3. *For each n,* $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$

Proof.

All of these properties hold for trigonometric polynomials, where convolution becomes multiplication under Fourier transform. Use that the trigonometric polynomials are dense with respect to integration (L^1 norm). □