Math 141: Lecture 20

Sequences and series of functions

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Products of series

Definition

Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two series. Their *Cauchy product* is the series

$$\sum_{n=0}^{\infty} c_n, \qquad c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

Example

The following example shows that it is possible that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge, but their Cauchy product $\sum_{n=0}^{\infty} c_n$ does not converge.

- Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}$. The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges by the alternating series test.
- The Cauchy product has $c_n = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}}$.
- By the Inequality of the Arithmetic Mean-Geometric Mean,

$$\sqrt{(k+1)(n-k+1)} \leq \frac{n+2}{2},$$

  and hence $|c_n| \geq \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2}$.
- Since $|c_n| \to 1$ as $n \to \infty$, the series does not converge.

Products of series

Theorem

Let $\sum_{n=0}^{\infty} a_n = A$ *converge absolutely, and* $\sum_{n=0}^{\infty} b_n = B$ *converge. Denote by* $\{c_n\}_{n=0}^{\infty}$ *the Cauchy product of* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$*. Then*

$$\sum_{n=0}^{\infty} c_n = AB.$$

Products of series

Proof of Cauchy Product formula. See Rudin, pp. 74-75.

- Define partial sums $A_n = \sum_{k=0}^n a_k$, $B_n = \sum_{k=0}^n b_k$, $C_n = \sum_{k=0}^n c_k$. Set $\beta_n = B_n - B$.
- Write

$$\begin{aligned} C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \\ &= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + \cdots a_n(B + \beta_0) \\ &= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0. \end{aligned}$$

- Define $\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0$.
- Since $A_n B \to AB$ as $n \to \infty$, it suffices to check that $\gamma_n \to 0$.

$\square$

Products of series

Proof of Cauchy Product formula. See Rudin, pp. 74-75.

- Recall that $\beta_n = B_n - B$ tends to 0 with n, and that we wish to show that $\gamma_n = a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0$ tends to 0, also.
- Define $\alpha = \sum_{n=0}^\infty |a_n|$.
- Given $\epsilon > 0$, choose N such that $n > N$ implies $|\beta_n| < \epsilon$.
- Bound

$$|\gamma_n| \le \sum_{k=0}^{N} |a_{n-k}\beta_k| + \sum_{k=N+1}^{n} |a_{n-k}\beta_k| \le \sum_{k=0}^{N} |a_{n-k}\beta_k| + \epsilon\alpha.$$

- Since $a_n \to 0$ as $n \to \infty$, if n is sufficiently large, then $\sum_{k=0}^N |a_{n-k}\beta_k| < \epsilon$, whence $|\gamma_n| \le (1+\alpha)\epsilon$. Letting $\epsilon \downarrow 0$ completes the proof.

$\square$

Example

The exponential function is often defined as the power series

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

which converges absolutely in the whole complex plane by the ratio test. We check that this series satisfies the expected multiplicative property.

$$\begin{aligned} E(z)E(w) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} \\ &= E(z+w). \end{aligned}$$

Example with order of limits

Consider the following:

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{m}{m+n}\right)=\lim_{m\to\infty}0=0$$
$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{m}{m+n}\right)=\lim_{n\to\infty}1=1.$$

Thus, the order in which limits are taken matters.

Pointwise convergence

Recall the definition of pointwise convergence.

Definition

A sequence of functions $\{f_n\}_{n=1}^{\infty}$ *converges pointwise* on a set E if, for each $x \in E$, $\lim_{n\to\infty} f_n(x)$ exists. The function f defined on E by

$$f(x) = \lim_{n\to\infty} f_n(x)$$

is called the *limit function*.

Uniform convergence

Recall that we have defined the supremum of a set which is bounded above to be the least upper bound. If a set is not bounded above, define its supremum to be ∞.

Definition

A sequence of functions $\{f_n\}_{n=1}^\infty$ *converges uniformly* on a set E if there exists a function f on E such that, for each $\epsilon > 0$ there exists $N > 0$ such that, for $n > N$,

$$\sup_{x \in E} |f(x) - f_n(x)| < \epsilon.$$

The sequence is said to be *uniformly Cauchy* on E if, for each $\epsilon > 0$ there exists $N > 0$ such that $m, n > N$ implies

$$\sup_{x \in E} |f_m(x) - f_n(x)| < \epsilon.$$

Example

Let $f_n(x) = \frac{x^2}{(1+x^2)^n}$ and set

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}.$$

Note that $f_n(0) = 0$ for all n, so $f(0) = 0$. For $x \neq 0$, summing the geometric series gives

$$f(x) = \frac{x^2}{1 - \frac{1}{1+x^2}} = 1 + x^2.$$

Thus the sum converges pointwise to a discontinuous function.

Example

Define for $m = 1, 2, ...,$

$$f_m(x) = \lim_{n\to\infty} \cos(m!\pi x)^{2n}.$$

This function is 1 if and only if $m!x$ is an integer. The function

$$\lim_{m\to\infty} f_m(x)$$

is 1 at rational x and 0 at irrational x, hence is not Riemann integrable.

Example

- Let $f_n(x) = \frac{\sin nx}{\sqrt{n}}$. This sequence of functions converges uniformly to 0.
- $f_n'(x) = \sqrt{n}\cos nx$. The sequence of derivatives does not converge even pointwise, for instance, at 0.
- This example shows that a condition stronger than uniform convergence is required to guarantee the convergence of derivatives.

Example

Let $f_n(x) = nx(1-x^2)^n$. For each $x \in [0,1]$, $\lim_{n\to\infty} f_n(x) = 0$. Thus

$$\int_0^1 \lim_{n\to\infty} f_n(x)dx = \int_0^1 0dx = 0.$$

On the other hand,

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} n\int_0^1 x(1-x^2)^n dx = \lim_{n\to\infty} \frac{n}{2n+2} = \frac{1}{2}.$$

Weierstrass M-test

Theorem (The Weierstrass M-test)

Let $\sum_{n=1}^{\infty} u_n$ be a series of functions which converges pointwise to a function u on a set S. Suppose that there are positive constants $\{M_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} M_n < \infty$ and such that

$$0 \leq |u_n(x)| \leq M_n$$

for all $x \in S$. Then $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on S.

Weierstrass M-test

Proof.

Given $\epsilon > 0$, choose N such that $\sum_{n=N+1}^{\infty} M_n < \epsilon$. For $M > N$ and $x \in S$, bound

$$\left| u(x) - \sum_{k=1}^{M} u_k(x) \right| = \left| \sum_{k=M+1}^{\infty} u_k(x) \right| \leq \sum_{k=M+1}^{\infty} |u_k(x)| \leq \sum_{k=M+1}^{\infty} M_k < \epsilon.$$

This proves the uniform convergence. □

Uniform convergence preserves continuity

Theorem

Let $x \in [a, b]$ and let $\{u_n\}_{n=1}^{\infty}$ be a sequence of functions which are continuous at x and converge uniformly on $[a, b]$ to a limit u. Then u is continuous at x.

We checked a statement similar to this in our discussion of the Weierstrass approximation theorem, but give a proof now in any case for completeness.

Uniform convergence preserves continuity

Proof.

- Given $\epsilon > 0$, choose N such that $n > N$ implies $\|u_n - u\|_\infty < \frac{\epsilon}{3}$.
- Choose $n > N$, and let $\delta > 0$ be such that $y \in [a, b]$ and $|x - y| < \delta$ implies $|u_n(x) - u_n(y)| < \frac{\epsilon}{3}$.
- Then,

$$\begin{aligned} |u(x) - u(y)| &= |(u(x) - u_n(x)) + (u_n(x) - u_n(y)) + (u_n(y) - u(y))| \\ &\le |u(x) - u_n(x)| + |u_n(x) - u_n(y)| + |u_n(y) - u(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

□

$C([a,b])$ is complete

Theorem

Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on $[a,b]$ which is uniformly Cauchy. Then $\{u_n\}_{n=1}^{\infty}$ converges uniformly to a continuous function u on $[a,b]$.

We say that the space $C([a,b])$ of continuous functions on $[a,b]$ is complete under the uniform distance.

$C([a,b])$ is complete

Proof.

- For each $x \in [a,b]$, the sequence $\{u_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers, and hence has a limit, call it $u(x)$.
- Given $\epsilon > 0$ choose $N > 0$ such that $m, n > N$ implies

$$|u_m(x) - u_n(x)| < \epsilon,$$

  uniformly in x.
- Taking the limit as $n \to \infty$,

$$|u_m(x) - u(x)| \leq \epsilon.$$

  Since this holds for all x, it implies the uniform convergence.
- The limit function u is continuous by the previous theorem.

$\square$

Uniform limits and integration

Theorem

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of integrable functions on $[a, b]$, converging uniformly to function f. Then f is integrable, and

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

Uniform limits and integration

Proof.

- Given $\epsilon > 0$, let N be such that $n > N$ implies $\|f - f_n\|_\infty < \epsilon$.
- Choose step functions s_n, t_n with $s_n < f_n < t_n$ and such that

$$\int_a^b t_n(x)dx - \epsilon < \int_a^b f_n(x)dx < \int_a^b s_n(x)dx + \epsilon.$$

- Notice that $s_n - \epsilon < f < t_n + \epsilon$, and the integrals of $s_n - \epsilon$ and $t_n + \epsilon$ differ by at most $2(1 + b - a)\epsilon$.
- Letting $\epsilon \downarrow 0$ proves that the lower and upper integrals of f are equal, so f is integrable. We also obtain that

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

$\square$

Uniform limits and differentiation

Theorem

Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}_{n=1}^{\infty}$ converges for some point $x_0 \in [a, b]$. If $\{f_n'\}_{n=1}^{\infty}$ converges uniformly on $[a, b]$, then $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[a, b]$, to a function f, and

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

Uniform limits and differentiation

Proof.

- Given $\epsilon > 0$, let N be such that $m, n > N$ implies $\|f_n' - f_m'\|_\infty < \frac{\epsilon}{2(b-a)}$ and $|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2}$.
- Given $x \in [a, b]$, by the Mean Value Theorem,

$$|f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0))| < \frac{|x - x_0|\epsilon}{2(b-a)} \leq \frac{\epsilon}{2}.$$

- Hence, by the triangle inequality,

$$\begin{aligned} |f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0))| \\ + |f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

- It follows that $\{f_n\}_{n=1}^\infty$ is uniformly Cauchy, hence uniformly convergent to a function f.

$\square$

Uniform limits and differentiation

Proof.

- Recall that, for $m, n > N$,

$$|f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0))| < \frac{|x - x_0|\epsilon}{2(b-a)} \leq \frac{\epsilon}{2}.$$

- Define $\phi_n(t) = \begin{cases} \frac{f_n(t)-f_n(x_0)}{t-x_0} & t \neq x_0 \\ f_n'(x_0) & t = x_0 \end{cases}.$
- $\phi_n(t)$ is continuous at x_0 for each n, and the sequence converges uniformly to a function $\phi(t)$, which is thus continuous at x_0.
- For $t \neq x_0$, $\phi(t) = \frac{f(t)-f(x_0)}{t-x_0}$. Hence $\phi(x_0) = f'(x_0) = \lim_{n\to\infty} f_n'(x_0)$.
- Since we've check that $f_n(x)$ converges for all $x \in [a, b]$, the choice of $x_0 \in [a, b]$ is arbitrary.

$\square$

Uniform convergence of power series

Theorem

Assume the power series $\sum_{n=0}^{\infty} a_n z^n$ *converges at* $z = z_1 \neq 0$. *Then*

1. *The series converges absolutely for every* z *with* $|z| < |z_1|$.
2. *The series converges uniformly on every circular disc of radius* $R < |z_1|$.

Proof.

Since $\sum a_n z_1^n$ converges, its terms tend to zero. Set $M = \max_n |a_n z_1^n|$. For $R < |z_1|$ and z such that $|z| \leq R$,

$$|a_n z^n| = |a_n z_1^n| \left|\frac{z}{z_1}\right|^n \leq M \left|\frac{R}{z_1}\right|^n.$$

The uniform and absolute convergence now follows from the Weierstrass M-test with $M_n = M \left|\frac{R}{z_1}\right|^n$. □

Differentiation of power series

Theorem

Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ *have radius of convergence* r*. Then for* $|x-a| < r$,

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$$

and this series has the same radius of convergence.

Differentiation of power series

Proof.

- To prove the theorem, it suffices to prove that $\sum_{n=1}^\infty na_n(x-a)^{n-1}$ converges absolutely on intervals $\{x : |x-a| < r_1\}$ for each $r_1 < r$, since then we can use the sequence of partial sums of $\sum_{n=0}^\infty a_n(x-a)^n$ in the theorem on sequences of differentiated functions.
- To check the uniform convergence, let $r_1 < r_2 < r$ and note that $|a_n|r_2^n$ is bounded, say by M.
- The Weierstrass M-test applies with $M_n = Mn\left(\frac{r_1}{r_2}\right)^n$. This series converges by the ratio test.

$\square$

Example

- Recall that the exponential function e^x satisfies the first order linear differential equation $f'(x) = f(x)$ and $f(0) = 1$.
- We verify that the series $E(x)$ satisfies this property.
  - $E(0) = \sum_{n=0}^{\infty} \frac{0^n}{n!} = 1$ (use the convention that the x^0 term is a constant)
  - $$E'(x) = \sum_{n=0}^{\infty} n\frac{x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = E(x).$$
- By the uniqueness theorem for first order linear ODE's this verifies that $e^x = E(x)$ for real x.

Example

Similarly, by the uniqueness theorem for second order linear ODE's, $\sin x$ is the unique solution of

$$f''(x) = -f(x), \qquad f(0) = 0,\ f'(0) = 1$$

and $\cos x$ is the unique solution of

$$f''(x) = -f(x), \qquad f(0) = 1,\ f'(0) = 0.$$

Define series

$$S(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad C(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Checking the differential equations proves that $S(x) = \sin x$ and $C(x) = \cos x$.

Euler's equation

Theorem

For all $x \in \mathbb{C}$, $e^{ix} = \cos x + i \sin x$.

Proof.

Write

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}.$$

Since the series is absolutely convergent, we can use $i^2 = -1$ and separate odd and even terms to obtain

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ &= \cos x + i \sin x. \end{aligned}$$

$\square$

Uniqueness of power series representations

Theorem

If two power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ and $\sum_{n=0}^{\infty} b_n(x-a)^n$ have the same sum function f in a neighborhood of the point a then for every n, $a_n = b_n$.

Proof.

We have $a_n = b_n = f^{(n)}(a)/n!$. $\square$

Integration of power series

Theorem

Let f be represented by a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

in the interval $(a-r, a+r)$*,* $r > 0$*. Then for* $|x-a| < r$*,*

$$\int_a^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}.$$

Proof.

This follows from the uniform convergence of the sequence of partial sums in the closed interval of radius $|x-a|$ about a. □