Math 141: Lecture 19

Convergence of infinite series

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Series of positive terms

Recall that, given a sequence $\{a_n\}_{n=1}^{\infty}$, we say that its series $\sum_{n=1}^{\infty} a_n$ converges if the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$

$$s_n = \sum_{k=1}^{n} a_k,$$

converges. We begin by giving some criteria for the convergence in the case when the terms a_n are non-negative.

Comparison test

Theorem (Comparison test)

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences such that $a_n \geq 0$ and $b_n \geq 0$ for all $n \geq 1$. Suppose there is a number N and a constant $c > 0$ such that

$$a_n \leq cb_n$$

for all $n > N$. Then convergence of $\sum_{n=1}^{\infty} b_n$ implies convergence of $\sum_{n=1}^{\infty} a_n$.

In this case, we say that the sequence b_n dominates the sequence a_n.

Comparison test

Proof of the comparison test.

- Suppose that $\sum_{n=1}^{\infty} b_n$ converges.
- Given $\epsilon > 0$, apply the Cauchy property of the sequence of partial sums of b_n to choose $M > N$ such that $m > n > M$ implies

$$\sum_{k=m+1}^{n} b_n < \frac{\epsilon}{c}.$$

- It follows that

$$\sum_{k=m+1}^{n} a_n \leq \sum_{k=m+1}^{n} cb_n < \epsilon,$$

so the sequence of partial sums of a_n is Cauchy, hence converges.

□

Limit comparison test

Theorem (Limit comparison test)

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences satisfying $a_n > 0$ and $b_n > 0$ for all $n \geq 1$. Suppose that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

Proof.

The condition implies that there is N such that $n > N$ implies

$$\frac{a_n}{2} \leq b_n \leq 2a_n.$$

It now follows from the comparison test that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges. □

Examples

- Recall that last class we checked

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1$$

  since the series telescopes. Since $\lim_{n\to\infty} \frac{n^2+n}{n^2} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1$, it follows that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.
- By comparison, it follows that $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for all $s \geq 2$.

Examples

- Recall that the harmonic series satisfies $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. One may check that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+10)}} = \infty, \qquad \sum_{n=1}^{\infty} \sin \frac{1}{n} = \infty$$

by noting that

$$\lim_{n\to\infty} \frac{\sqrt{n(n+10)}}{n} = \lim_{n\to\infty} \sqrt{1 + \frac{10}{n}} = 1,$$

and

$$\lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} n\left(\frac{1}{n} + O\left(\frac{1}{n^3}\right)\right) = \lim_{n\to\infty} 1 + O\left(\frac{1}{n^2}\right) = 1.$$

Integral test

Theorem (Integral test)

Let f be a positive decreasing function, defined for all real $x \geq 1$. The infinite series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the improper integral $\int_1^{\infty} f(x)dx$ converges.

Integral test

Proof of the integral test.

- We first check that the improper integral converges if and only if the sequence $\left\{t_n = \int_1^n f(x)dx\right\}_{n=1}^{\infty}$ converges.
- Recall that convergence of the improper integral is defined by the existence of the limit $\lim_{x\to\infty} \int_1^x f(t)dt$.
- Evidently convergence of the integral entails convergence of the sequence as a special case.
- To check the reverse direction, note that for $n \leq x \leq n+1$, $t_n \leq \int_1^x f(t)dt \leq t_{n+1}$, and hence, if $|t_n - L| < \epsilon$ for all $n \geq N$, then

$$\left|\int_1^x f(t)dt - L\right| < \epsilon$$

for all $x \geq N$ also.

$\square$

Integral test

Proof of the integral test.

- We now show that $\{t_n\}_{n=1}^\infty$ converges if and only if the sequence of partial sums $\{s_n = \sum_{k=1}^n f(k)\}_{k=1}^\infty$ converges.
- Given $m > n$, note that by taking upper and lower step functions for the integral,

$$\sum_{k=n}^{m-1} f(k) \geq t_m - t_n = \int_n^m f(t)dt \geq \sum_{k=n+1}^{m} f(k)$$

and hence $\{t_n\}_{n=1}^\infty$ is Cauchy if and only if $\{s_n\}_{n=1}^\infty$ is Cauchy, which proves the claim.

$\square$

Examples

- We can show that the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$, s real, converges if and only if $s > 1$. To check this, note that

$$t_n = \int_1^n \frac{dt}{t^s} = \begin{cases} \frac{n^{1-s}-1}{1-s} & s \neq 1 \\ \log n & s = 1 \end{cases}.$$

  This sequence converges if and only if $s > 1$.

- For complex $s = \sigma + it$, $\frac{1}{n^s} = \frac{1}{n^\sigma} \exp(it \log n)$. Since $|\exp(it \log n)| = 1$, the sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

  converges absolutely if $\sigma = \Re(s) > 1$. This function is called the Riemann zeta function.

Examples

The study of the analytic properties of the Riemann zeta function and related functions is responsible for a great deal of the information that we know about the prime numbers.

Root test

Theorem (Root test)

Let $\sum_{n=1}^{\infty} a_n$ be a series of non-negative terms. Suppose that

$$a_n^{\frac{1}{n}} \to R, \qquad n \to \infty.$$

If $R < 1$ then the series converges. If $R > 1$ then the series diverges.

Root test

Proof of the root test.

Let $x = \frac{R+1}{2}$ so that $R < x < 1$ if $x < 1$ and $1 < x < R$ if $R > 1$. Suppose $R < 1$. Then, for all n sufficiently large,

$$a_n < x^n.$$

The series converges by comparison with the geometric series $\sum_{n=1}^{\infty} x^n$. Suppose instead that $R > 1$. Then for all n sufficiently large,

$$a_n > x^n$$

so that the series diverges by comparison with the geometric series $\sum_{n=1}^{\infty} x^n$. $\square$

Example

- We apply the root test to the series $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$. Calculate

$$a_n^{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+\frac{1}{n}}\right)^n \to \frac{1}{e}.$$

Thus the series converges.

Ratio test

Theorem

Let $\sum_{n=1}^{\infty} a_n$ *be a sum of positive terms such that*

$$\frac{a_{n+1}}{a_n} \to L, \qquad n \to \infty.$$

The series converges if $L < 1$ *and diverges if* $L > 1$*.*

Ratio test

Proof of the ratio test.

If $L > 1$ then the series does not converge since the terms do not tend to 0. If $L < 1$, let

$$L < x = \frac{L+1}{2} < 1.$$

Choose N sufficiently large, so that if $n \geq N$ then $\frac{a_{n+1}}{a_n} < x$. It follows that, for all $n > N$,

$$a_n = a_N \prod_{k=N}^{n-1} \frac{a_{k+1}}{a_k} \leq a_N x^{n-N} = \frac{a_N}{x^N} x^n.$$

Since $\frac{a_N}{x^N}$ is just a constant, the series converges by comparison with the geometric series $\sum_{n=1}^{\infty} x^n$. $\square$

Examples

- Consider the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$. Applying the ratio test, as $n \to \infty$,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n = \frac{1}{(1+1/n)^n} \to \frac{1}{e}.$$

  Hence the series converges.

Edge cases

The ratio and root tests are not useful in the case $\frac{a_{n+1}}{a_n} \to 1$ as $n \to \infty$. Note that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ while $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. The ratios of consecutive terms in these cases are given by

$$\frac{n}{n+1} = 1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right), \qquad \frac{n^2}{(n+1)^2} = 1 - \frac{2}{n} + O\left(\frac{1}{n^2}\right).$$

The following two tests of Raabe and Gauss sometimes help to decide convergence in cases where the limiting ratio is 1.

Raabe's test

Theorem (Raabe's test)

Let $\sum_{n=1}^\infty a_n$ be a sum of positive terms. If there is an $r > 0$ and $N \geq 1$ such that, for all $n \geq N$,

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{1}{n} - \frac{r}{n}$$

then $\sum_{n=1}^\infty a_n$ converges. If, for all $n \geq N$,

$$\frac{a_{n+1}}{a_n} \geq 1 - \frac{1}{n}$$

then $\sum_{n=1}^\infty a_n = \infty$.

An inequality for the logarithm

In the first part of the proof of Raabe's test we use the inequality

$$\log(1 + x) \leq x, \qquad -1 < x.$$

To check this inequality, note equality holds at $x = 0$. Also, $x - \log(1 + x)$ has derivative $1 - \frac{1}{1+x}$, which changes sign at $x = 0$. It follows that the function decreases to 0, and increases thereafter.

Raabe's test

Proof of Raabe's test.

Assume first that $\frac{a_{n+1}}{a_n} \leq 1 - \frac{1+r}{n}$ for $n > N$. Thus, for $n > N$,

$$a_n \leq a_N \prod_{k=N}^{n-1} \left(1 - \frac{r+1}{k}\right) \leq a_N \exp\left(-\sum_{k=N}^{n-1} \frac{r+1}{k}\right).$$

There is some constant C (which depends on N) such that

$$\sum_{k=N}^{n-1} \frac{1}{k} \geq -C + \log n.$$

Hence, there is another constant c, depending on N, such that $a_n \leq \frac{c}{n^{1+r}}$ for $n > N$. The series thus converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^{1+r}}$. $\square$

Raabe's test

Proof of Raabe's test.

Now assume that $\frac{a_{n+1}}{a_n} \geq 1 - \frac{1}{n}$ for $n > N \geq 1$. Thus, for $n > N$,

$$a_n \geq a_N \prod_{k=N}^{n-1} \left(1 - \frac{1}{k}\right).$$

Note that the product is equal to $\prod_{k=N}^{n-1} \frac{k-1}{k} = \frac{N-1}{n-1}$, since it telescopes. Since there is a constant c (depending on N) such that $a_n > \frac{c}{n-1}$ for all $n > N$, the series diverges by comparison with the harmonic series. □

Gauss's test

Theorem (Gauss's test)

Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms. If there is an $N \geq 1$, an $s > 1$ and an $M > 0$ such that for all $n > N$,

$$\frac{a_{n+1}}{a_n} = 1 - \frac{A}{n} + \frac{f(n)}{n^s}$$

where $|f(n)| \leq M$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges if $A > 1$ and diverges if $A \leq 1$.

Gauss's test

Proof of Gauss's test.

Let N be sufficiently large, and write, for $n > N$,

$$a_n = a_N \prod_{k=N}^{n-1} \frac{a_{k+1}}{a_k} = a_N \prod_{k=N}^{n-1} \left(1 - \frac{A}{k} + \frac{f(k)}{k^s}\right).$$

Since $\frac{A}{k} - \frac{f(k)}{k^s}$ tends to 0 as $k \to \infty$ we may write $\log\left(1 - \frac{A}{k} + \frac{f(k)}{k^s}\right) = -\frac{A}{k} + \frac{f(k)}{k^s} + O\left(\frac{1}{k^2}\right)$. Hence

$$\begin{aligned} a_n &= a_N \exp\left(\sum_{k=N}^{n-1}\left(\frac{-A}{k} + \frac{f(k)}{k^s} + O\left(\frac{1}{k^2}\right)\right)\right) \\ &= a_N \exp\left(-A \log n + O(1)\right). \end{aligned}$$

The series thus converges or diverges according as $A > 1$ or otherwise, by comparison with the sum $\sum_n \frac{1}{n^A}$. □

Examples

- Consider the series $\sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right)^k$. The ratio $\frac{a_n}{a_{n-1}}$ of this sequence is given by

$$\left(\frac{2n-1}{2n}\right)^k = \left(1-\frac{1}{2n}\right)^k$$
$$= \exp\left(k\left(-\frac{1}{2n}+O\left(\frac{1}{n^2}\right)\right)\right)$$
$$= 1-\frac{k}{2n}+O\left(\frac{1}{n^2}\right)$$

as $n\to\infty$. Hence the series converges for $k>2$ and diverges for $k\leq 2$ by Gauss's test.

Power series

Definition

Let $\{a_n\}_{n=0}^\infty$. The series $P(x) = \sum_{n=0}^\infty a_n x^n$ is called the *power series* associated to $\{a_n\}_{n=0}^\infty$.

The series $\sum_{n=0}^\infty a_n x^n$ is sometimes also called a (linear) generating function of $\{a_n\}_{n=0}^\infty$. We'll return to study power series in more detail in the next several lectures, but point out some basic properties now.

Power series

Theorem

Let $\{a_n\}_{n=0}^{\infty}$ *be a sequence. Define* $R = 0$ *if* $\{|a_n|^{\frac{1}{n}}\}_{n=1}^{\infty}$ *is unbounded,* $R = \infty$ *if* $|a_n|^{\frac{1}{n}} \to 0$*, and*

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

otherwise. The power series $P(x) = \sum_{n=0}^{\infty} a_n x^n$ *converges absolutely in the disc* $D_R(0) = \{x \in \mathbb{C} : |x| < R\}$ *and diverges for* $|x| > R$*.*

Definition

As defined, R is called the *radius of convergence* of the power series $P(x) = \sum_{n=0}^{\infty} a_n x^n$.

Power series

Proof.

- First suppose $R = 0$, so that $|a_n|^{\frac{1}{n}}$ is unbounded. It follows that for any $x \neq 0$, $|a_n x^n|^{\frac{1}{n}} = |a_n|^{\frac{1}{n}}|x|$ is unbounded. The power series $P(x)$ diverges since its terms do not tend to 0.
- Next suppose $R = \infty$ so that $|a_n|^{\frac{1}{n}} \to 0$. Then for any $x \in \mathbb{C}$, $|a_n x^n|^{\frac{1}{n}} = |a_n|^{\frac{1}{n}}|x| \to 0$, so that the power series converges absolutely, by comparison with a geometric series.
- Finally, suppose that $R > 0$ is finite. For $|x| < R$,

$$\limsup_{n\to\infty} |a_n x^n|^{\frac{1}{n}} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}|x| = \frac{|x|}{R} < 1.$$

  The series thus converges absolutely by comparison with a geometric series. If instead $|x| > R$, then the limsup is $\frac{|x|}{R} > 1$, and the power series does not converge since its terms do not tend to 0.

$\square$

Solving linear recurrences

A *linear recurrence sequence* is a sequence in which successive terms are defined as a linear combination of a bounded number of terms preceeding them (and possibly an auxiliary function). Examples include

- The Fibonacci numbers, $F_0 = F_1 = 1$, for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.
- The number of steps needed to solve the Towers of Hanoi with n discs, $T_0 = 0$, $T_{n+1} = 2T_n + 1$. $(T_n = 2^n - 1)$.
- The number of regions into which the plane is split by n lines in general position (pairwise non-parallel), $L_0 = 1$, $L_{n+1} = L_n + n + 1$. $(L_n = 1 + \binom{n+1}{2})$.

Solving linear recurrences

Although there are other methods, often the easiest way of solving linear recurrences is through writing down the power series generating function.

- Fibonacci numbers: Let $f(x) = \sum_{n=0}^{\infty} F_n x^n$. By the recurrence relation,

$$\begin{aligned} f(x) &= 1 + x + \sum_{n=2}^{\infty} F_n x^n \\ &= 1 + x + \sum_{n=2}^{\infty} (F_{n-2} + F_{n-1}) x^n \\ &= 1 + (x + x^2) f(x). \end{aligned}$$

  Hence $f(x)(1 - x - x^2) = 1$, or $f(x) = \frac{1}{1-x-x^2}$.

Solving linear recurrences

Recall $f(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{1}{1-x-x^2}$.

- Let $\phi = \frac{1+\sqrt{5}}{2}$, $\frac{1}{\phi} = \frac{\sqrt{5}-1}{2}$ so that $(1-x-x^2) = (1-\phi x)(1+\frac{x}{\phi})$.
- Write in partial fractions,

$$\frac{1}{1-x-x^2} = \frac{A}{1-\phi x} + \frac{B}{1+\frac{x}{\phi}}$$

  with $A + B = 1$, $\frac{A}{\phi} - B\phi = 0$ so that $A = \frac{\phi^2}{1+\phi^2}$, $B = \frac{1}{1+\phi^2}$.
- Matching coefficients of x^n,

$$F_n = \frac{\phi^2}{1+\phi^2}\phi^n + \frac{(-1)^n}{1+\phi^2}\phi^{-n}.$$

- The radius of convergence of the power series $f(x)$ is $\frac{1}{\phi}$.

Partial summation

Theorem (Abel summation)

Let $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two sequences of complex numbers, and let*

$$A_n = \sum_{k=1}^{n} a_k.$$

Then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k(b_k - b_{k+1}).$$

Partial summation

Proof.

Define $A_0 = 0$. Thus, since $a_k = A_k - A_{k-1}$, $k = 1, 2, \ldots$, one has

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= \sum_{k=1}^{n} (A_k - A_{k-1}) b_k \\ &= \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_k b_{k+1} + A_n b_{n+1} \\ &= A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}).\end{aligned}$$

$\square$

Dirichlet's test

Theorem (Dirichlet's test)

Let $\{a_n\}$ be a sequence of complex terms, whose partial sums form a bounded sequence. Let $\{b_n\}$ be a decreasing sequence of positive real terms, which tends to 0. Then the series

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

This generalizes the alternating series test from last lecture.

Dirichlet's test

Proof of Dirichlet's test.

Denote the sequence of partial sums of $\{a_n\}_{n=1}^\infty$ by $\{A_n\}_{n=1}^\infty$. Let $M > 0$ be such that $|A_n| \leq M$ for all n. By Abel summation, the sequence of partial sums $\{s_n = \sum_{k=1}^n a_k b_k\}_{n=1}^\infty$ satisfy

$$s_n = A_n b_{n+1} + \sum_{k=1}^n A_k(b_k - b_{k+1}).$$

Note that $A_n b_{n+1} \to 0$ as $n \to \infty$, while $\sum_{k=1}^\infty A_k(b_k - b_{k+1})$ converges absolutely, since

$$\sum_{k=1}^\infty |A_k(b_k - b_{k+1})| \leq M \sum_{k=1}^\infty (b_k - b_{k+1}) = Mb_1.$$

This proves the convergence. $\square$

Examples

- Let $\theta \in \mathbb{R}$. The sum $\sum_{k=1}^{\infty} \frac{e^{2\pi i k\theta}}{k}$ converges if θ is not an integer and diverges otherwise.
- To see this, for $\theta \in \mathbb{Z}$, $e^{2\pi i k\theta} = 1$, so this is the harmonic series, which diverges.
- Otherwise, set $z = e^{2\pi i\theta}$ so $|z| = 1$, $z \neq 1$. Then $A_n = \sum_{k=1}^{n} e^{2\pi i k\theta} = \frac{z - z^n}{1-z}$ satisfies $|A_n| \leq \frac{2}{|1-z|}$. Hence the convergence follows from Dirichlet's test.

Abel's test

Theorem (Abel's test)

Let $\sum_{n=1}^\infty a_n$ *be a convergent complex series, and let* $\{b_n\}_{n=1}^\infty$ *be a monotonic, convergent sequence of real terms. Then*

$$\sum_{n=1}^\infty a_n b_n$$

converges.

Abel's test

Proof of Abel's test.

As in the proof of Dirichlet's test, write $A_n = \sum_{k=1}^n a_k$, and assume that $|A_n| \leq M$. Let $b = \lim_{n\to\infty} b_n$. By Abel summation,

$$\sum_{k=1}^n a_k b_k = A_n b_{n+1} + \sum_{k=1}^n A_k(b_k - b_{k+1}).$$

As $n \to \infty$, $\lim_{n\to\infty} A_n b_{n+1} = b\sum_{k=1}^\infty a_k$ converges. Also,

$$\sum_{k=1}^\infty |A_k(b_k - b_{k+1})| \leq M \sum_{k=1}^\infty |b_k - b_{k+1}| = M|b_1 - b|$$

since the series on the right telescopes. Thus $\sum_{k=1}^\infty A_k(b_k - b_{k+1})$ converges absolutely. $\square$

Products of series

Definition

Let $\sum_{n=0}^\infty a_n$ and $\sum_{n=0}^\infty b_n$ be two series. Their *Cauchy product* is the series

$$\sum_{n=0}^\infty c_n, \qquad c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

Example

The following example shows that it is possible that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, but their Cauchy product $\sum_{n=1}^{\infty} c_n$ does not converge.

- Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}$. The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges by the alternating series test.
- The Cauchy product has $c_n = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}}$.
- By the Inequality of the Arithmetic Mean-Geometric Mean,

$$\sqrt{(k+1)(n-k+1)} \leq \frac{n+2}{2},$$

  and hence $|c_n| \geq \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2}$.
- Since $|c_n| \to 1$ as $n \to \infty$, the series does not converge.

Products of series

Theorem

Let $\sum_{n=0}^{\infty} a_n = A$ *converge absolutely, and* $\sum_{n=0}^{\infty} b_n = B$ *converge. Denote by* $\{c_n\}_{n=0}^{\infty}$ *the Cauchy product of* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$*. Then*

$$\sum_{n=0}^{\infty} c_n = AB.$$

Products of series

Proof of Cauchy Product formula. See Rudin, pp. 74-75.

- Define partial sums $A_n = \sum_{k=0}^n a_n$, $B_n = \sum_{k=0}^n b_k$, $C_n = \sum_{k=0}^n c_k$. Set $\beta_n = B_n - B$.
- Write

$$\begin{aligned} C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \\ &= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + \cdots a_n(B + \beta_0) \\ &= A_n B + a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0. \end{aligned}$$

- Define $\gamma_n = a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0$.
- Since $A_n B \to AB$ as $n \to \infty$, it suffices to check that $\gamma_n \to 0$.

$\square$

Products of series

Proof of Cauchy Product formula. See Rudin, pp. 74-75.

- Recall that $\beta_n = B_n - B$ tends to 0 with n, and that we wish to show that $\gamma_n = a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0$ tends to 0 also.
- Define $\alpha = \sum_{n=0}^{\infty} |a_n|$.
- Given $\epsilon > 0$, choose N such that $n > N$ implies $|\beta_n| < \epsilon$.
- Bound

$$|\gamma_n| \leq \sum_{k=0}^{N} |a_{n-k}\beta_k| + \sum_{k=N+1}^{n} |a_{n-k}\beta_k| \leq \sum_{k=0}^{N} |a_{n-k}\beta_k| + \epsilon\alpha.$$

- Since $a_n \to 0$ as $n \to \infty$, if n is sufficientlly large, then $\sum_{k=0}^{N} |a_{n-k}\beta_k| < \epsilon$, whence $|\gamma_n| \leq (1+\alpha)\epsilon$. Letting $\epsilon \downarrow 0$ completes the proof.

□