Math 141: Lecture 15

Stirling's approximation and Newton's method

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Stirling's approximation

Theorem

For integer $n \geq 1$, as $n \to \infty$

$$\log n! = \left(n + \frac{1}{2}\right) \log n - n + \frac{1}{2} \log 2\pi + O(1/n).$$

Stirling's approximation

Proof.

By the additive property of the logarithm, $\log n! = \sum_{j=1}^n \log j$. Denote $[x]$ the integer nearest to x, that is, $[x] = n$ if $x \in \left(n - \frac{1}{2}, n + \frac{1}{2}\right]$. Thus

$$\begin{aligned}\log n! &= \int_{\frac{1}{2}}^{n+\frac{1}{2}} \log[x] dx \\ &= \int_{\frac{1}{2}}^{n+\frac{1}{2}} \log x dx + \int_{\frac{1}{2}}^{n+\frac{1}{2}} (\log[x] - \log x) dx\end{aligned}$$

Write $\log[x] - \log x = -\log \frac{x}{[x]} = -\log\left(1 + \frac{x-[x]}{[x]}\right)$ to obtain

$$\left(n + \frac{1}{2}\right) \log\left(n + \frac{1}{2}\right) - \frac{1}{2} \log \frac{1}{2} - n - \int_{\frac{1}{2}}^{n+\frac{1}{2}} \log\left(1 + \frac{x - [x]}{[x]}\right) dx.$$

$\square$

Stirling's approximation

Proof.

Recall

$$\log n! = \left(n+\frac{1}{2}\right)\log\left(n+\frac{1}{2}\right) - \frac{1}{2}\log\frac{1}{2} - n - \int_{\frac{1}{2}}^{n+\frac{1}{2}} \log\left(1+\frac{x-[x]}{[x]}\right)dx.$$

Write

$$\log\left(n+\frac{1}{2}\right) = \log n + \log\left(1+\frac{1}{2n}\right) = \log n + \frac{1}{2n} + O\left(1/n^2\right).$$

Thus the first line is $\left(n+\frac{1}{2}\right)\log n - n + \frac{1}{2}\log 2 + \frac{1}{2}$. $\square$

Stirling's approximation

Proof.

Write the integral as

$$\int_{\frac{1}{2}}^{n+\frac{1}{2}} \log\left(1+\frac{x-[x]}{[x]}\right) dx = \sum_{j=1}^{n} \int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left(1+\frac{x}{j}\right) dx$$
$$= \frac{1}{2}\sum_{j=1}^{n} \int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left(1-\frac{x^2}{j^2}\right) dx.$$

The integral is $O\left(\frac{1}{j^2}\right)$. Hence

$$\int_{n+\frac{1}{2}}^{\infty} \log\left(1+\frac{x-[x]}{[x]}\right) dx = O\left(\int_{n}^{\infty} \frac{dx}{x^2}\right) = O(1/n).$$

The integral is thus $\int_{\frac{1}{2}}^{\infty} \log\left(1+\frac{x-[x]}{[x]}\right) dx + O(1/n)$.

□

Stirling's approximation

The previous slides prove $\log n! = \left(n + \frac{1}{2}\right) \log n - n + C + O(1/n)$ for some undetermined constant C.

Theorem

We have

$$\lim_{n\to\infty} \frac{\sqrt{n}\binom{2n}{n}}{2^{2n}} = \sqrt{\frac{1}{\pi}}.$$

To deduce the value of C from this theorem, note that

$$\begin{aligned}
\log \binom{2n}{n} &= \log(2n)! - 2\log n! \\
&= \left(2n + \frac{1}{2}\right) \log 2n - (2n+1)\log n - C + O(1/n) \\
&= \left(2n + \frac{1}{2}\right) \log 2 - \frac{1}{2} \log n - C + O(1/n)
\end{aligned}$$

Letting $n \to \infty$ obtains $C - \frac{1}{2}\log 2 = \frac{1}{2}\log \pi$.

Stirling's approximation

Lemma

For each $n \geq 1$ we have

$$\frac{n!n!}{(2n+1)!} = \frac{1}{(2n+1)\binom{2n}{n}} = \int_0^1 x^n(1-x)^n dx.$$

Proof.

- Drop $2n+1$ points in the unit interval $[0,1]$ uniformly, independently at random. We claim that both sides of the equation give the probability of the event that the first n points dropped lie to the left of the $(n+1)$st point, and the last n points dropped lie to the right of the $(n+1)$st point.

$\square$

Stirling's approximation

Proof.

- There are $(2n+1)!$ orderings with which the points can appear, each of which is equally likely. There are $n!$ orderings of the first n points, $n!$ orderings of the last n, so $(n!)^2$ orderings which are permissible. This gives a probability of $\frac{(n!)^2}{(2n+1)!}$.
- When the $(n+1)$st point is at position x, the chance that the first n points lie to the left of it is x^n, with the last n to the right, $(1-x)^n$. Thus the ordering has chance $x^n(1-x)^n$. Averaging over all possible x, the probability is

$$\int_0^1 x^n(1-x)^n dx.$$

□

Stirling's approximation

Proof sketch of limit.

Make a change of variables to write

$$\begin{aligned}\int_0^1 x^n(1-x)^n dx &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(x+\frac{1}{2}\right)^n \left(\frac{1}{2}-x\right)^n dx\\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{4}-x^2\right)^n dx\\ &= \frac{1}{2^{2n}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp\left(n\log(1-4x^2)\right) dx\\ &= \frac{1}{2^{2n}\sqrt{n}} \int_{-\frac{\sqrt{n}}{2}}^{\frac{\sqrt{n}}{2}} \exp\left(n\log\left(1-\frac{4x^2}{n}\right)\right) dx.\end{aligned}$$

$\square$

Stirling's approximation

Proof sketch of limit.

The integral is well approximated by replacing $\log(1-\frac{4x^2}{n})$ with $-\frac{4x^2}{n}$ and then extending to $(-\infty,\infty)$ (this is a little technical) which gives

$$\begin{aligned} 2^{2n}\sqrt{n}\int_0^1 x^n(1-x)^n dx &= \int_{-\infty}^{\infty} \exp(-4x^2)dx + o(1) \\ &= \frac{1}{\sqrt{8}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx + o(1) = \frac{\sqrt{\pi}}{2} + o(1). \end{aligned}$$

It follows that

$$\lim_{n\to\infty} 2^{2n}\sqrt{n}\frac{(n!)^2}{(2n+1)!} = \frac{\sqrt{\pi}}{2}$$

and thus

$$\lim_{n\to\infty} \frac{\sqrt{n}\binom{2n}{n}}{2^{2n}} = \lim_{n\to\infty} \frac{2}{\sqrt{\pi}}\frac{n}{2n+1} = \frac{1}{\sqrt{\pi}}.$$

$\square$

Calculating π

Various methods of calculating the decimal expansion of π use the Taylor series of $\arctan x$ at small angles x.

Theorem

If $0 \leq x \leq 1$ *then for each* $n \geq 1$*,*

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - ... + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + E_n(x)$$

with $|E_n(x)| \leq \frac{x^{2n+1}}{2n+1}$.

Proof.

Write $\frac{1}{1+x^2} = 1 - x^2 + x^4 - ... + (-1)^{n-1}x^{2n-2} + \frac{(-1)^n x^{2n}}{1+x^2}$. Integrating,

$$|E_n(x)| = \int_0^x \frac{u^{2n}}{1+u^2} du \leq \int_0^x u^{2n} du = \frac{x^{2n+1}}{2n+1}.$$

$\square$

Calculating π

Theorem (John Machin (1680-1751))

$\pi = 16 \arctan \frac{1}{5} - 4 \arctan \frac{1}{239}$.

Proof.

Let $\alpha = \arctan \frac{1}{5}$ and $\beta = 4\alpha - \frac{\pi}{4}$. Thus $\pi = 16\alpha - 4\beta$, so it suffices to check that $\tan \beta = \frac{1}{239}$. The trig identity

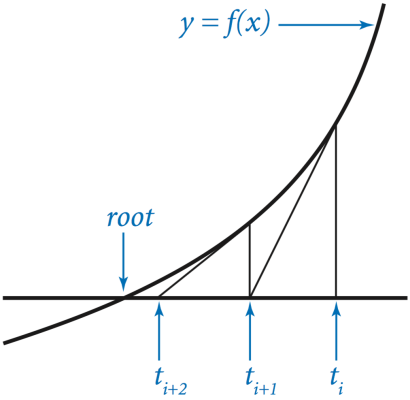
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

gives $\tan 2\alpha = \frac{5}{12}$, $\tan 4\alpha = \frac{120}{119}$ and $\tan\left(4\alpha - \frac{\pi}{4}\right) = \frac{1}{239}$. $\square$

Calculating π using this expansion gives at least a factor 25 improvement with each additional term in the Taylor expansion. One obtains a number of bits linear in the number of terms in the Taylor expansion.

Newton's method

Newton's method gives a method of numerically approximating a root of a differentiable curve.



Newton's method

- Newton's method approximates a root $f(r) = 0$ of a differentiable function f by making a series of guesses.
- Given an initial point x_0, each successive point x_{n+1} after x_n is found by making a linear approximation to $f(x)$ through the point $(x_n, f(x_n))$,
$$\ell_n(x) = f(x_n) + (x - x_n)f'(x_n).$$
- If $f'(x_n) \neq 0$, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ is the root of the linear approximation to f through x_n.

Newton's method

Theorem

Let f be twice continuously differentiable on an interval $[a, b]$ and assume $f(c) = 0$ for some $a < c < b$. Suppose that $f'(c) \neq 0$. Then there exists $\delta > 0$ with $\delta < \min(c - a, b - c)$ and $M > 0$ such that, for any $x_0 \in (c - \delta, c + \delta)$ the sequence $\{x_n\}_{n=0}^{\infty}$ defined recursively by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

satisfies, for all $n \geq 0$, $|x_{n+1} - c| \leq M|x_n - c|^2$.

Newton's method

- We say that the rate of convergence in Newton's method is quadratic, which means that the number of significant figures in an approximation essentially doubles on each iteration. In particular, the number of determined bits is exponential in the number of iterations. This makes Newton's method preferrable to Taylor expansion.
- The extremely fast convergence of Newton's method makes it a favorite algorithm in numerical analysis.

Newton's method

Proof.

- Let $\delta > 0$ be such that $|x - c| < \delta$ implies $|f'(x)| > \frac{|f'(c)|}{2}$.
- Let $m = \max_{x \in [c-\delta, c+\delta]} |f''(x)|$.
- Suppose $x_n \in (c - \delta, c + \delta)$. Write

$$x_{n+1} - c = x_n - c - \frac{f(x_n)}{f'(x_n)}.$$

□

Newton's method

Proof.

Recall $x_{n+1} - c = x_n - c - \frac{f(x_n)}{f'(x_n)}$.

- Taylor expand $f(x_n) = 0 + f'(c)(x_n - c) + \int_c^{x_n} (x_n - t) f''(t)dt$. Hence

$$x_{n+1} - c = (x_n - c)\left(1 - \frac{f'(c)}{f'(x_n)}\right) + \epsilon_1$$

  where $\epsilon_1 = \frac{1}{f'(x_n)} \int_c^{x_n} (x_n - t) f''(t)dt$ satisfies $|\epsilon_1| \leq \frac{M}{|f'(c)|}(x_n - c)^2$.
- By the Mean Value Theorem

$$\left|1 - \frac{f'(c)}{f'(x_n)}\right| = \left|\frac{f'(x_n) - f'(c)}{f'(x_n)}\right| \leq \frac{m|x_n - c|}{|f'(x_n)|} \leq \frac{2m|x_n - c|}{|f'(c)|}.$$

□

Newton's method

Proof.

- It follows that, under the assumption that $x_n \in (c-\delta, c+\delta)$,

$$|x_{n+1} - c| \leq M|x_n - c|^2, \qquad M = \frac{3m}{|f'(c)|}.$$

- Decrease δ sufficiently so that $M\delta^2 < \delta$, or $M < \frac{1}{\delta}$. Then $x_n \in (c-\delta, c+\delta)$ implies $x_{n+1} \in (c-\delta, c+\delta)$ so $\{x_n\}_{n=0}^{\infty} \subset (c-\delta, c+\delta)$.
- Thus, for all n, $|x_{n+1} - c| \leq M|x_n - c|^2$ holds.

$\square$

Newton's method example

We approximate $\sqrt{2}$ by applying Newton's method to the equation $f(x) = x^2 - 2 = 0$ with initial guess $x_0 = 1$.

- Note that $f'(x) = 2x$ and hence $x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$.
- Hence $x_1 = 1.5$, $x_2 = 1.41\overline{6}$, $x_3 = 1.4142156862(7)$, $x_4 = 1.4142135623(7)$.
- We have $\sqrt{2} = 1.41421356237...$ so 4 iterations give this precision.

Newton's method divergence

Suppose one attempts to find the root $x = 0$ of $f(x) = x^{\frac{1}{3}}$ by Newton's method. Note that f is not differentiable at 0.

- For $x \neq 0$, $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$.
- Hence, given a guess $x_0 \neq 0$,

$$x_1 = x_0 - \frac{x_0^{\frac{1}{3}}}{\frac{1}{3}x_0^{-\frac{2}{3}}} = -2x_0.$$

- Thus the sequence of guesses diverges exponentially away from the root.

Newton's method oscillation

- The polynomial $P(x) = (x-1)^2(x+1) + x^2(2-x) + x^4(x-1)^3$ satisfies $P(0) = P(1) = 1$ and $P'(0) = -1$, $P'(1) = 1$.
- Hence, started from an initial guess of either 0 or 1, the guesses oscillate between 0 and 1.

Polynomial equations on $\mathbb{C}$

Newton's method may be applied to find the roots of complex functions also. The following figure is colored according to the root of $x^5 - 1$ which an initial guess converges.