# Math 141: Lecture 14

## More limits, the Weierstrass approximation theorem, the Gaussian

Bob Hough

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# Taylor expansion of composites

## Problem

*Give the first six terms in the Taylor expansion of* $\sin(\cos x)$ *about* $x = 0$.

## Solution

*Since* $\cos 0 = 1$, *Taylor expand* $\sin$ *about 1 to find*

$$\sin(1+u) = \sin 1 + u\cos 1 - \frac{u^2}{2}\sin 1 - \frac{u^3}{6}\cos 1 + O(u^4).$$

*Set* $u = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + O(x^8)$. *Hence*

$$\sin(\cos x) = \sin 1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right)\cos 1 - \frac{\left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right)^2}{2}\sin 1 + \frac{x^6}{48}\cos 1 + O(x^8).$$

# Taylor expansion of composites

## Problem

*Give the first four terms in the Taylor expansion of* $e^{\sin x}$ *about* $x = 0$.

## Solution

*Use* $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + O(u^5)$ *with* $u = x - \frac{x^3}{3} + O(x^5)$ *to obtain*

$$e^{\sin x} = 1 + \left(x - \frac{x^3}{3!}\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!}\right)^2 + \frac{x^3}{3!} + \frac{x^4}{4!} + O(x^5).$$

# Exponential limits

## Problem

*Prove* $\lim_{x\to 0+} x^x = 1$.

## Solution

*By continuity of the exponential function at 0,*

$$\lim_{x\to 0+} x^x = \exp\left(\lim_{x\to 0+} x\log x\right).$$

*Write* $x\log x = \frac{\log x}{\frac{1}{x}}$ *and apply l'Hôpital's Rule to conclude this limit is 0. Thus*

$$\lim_{x\to 0+} x^x = 1.$$

# Exponential limits

## Theorem

*The function*

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

*is infinitely differentiable and has all derivatives at 0 equal to 0.*

## Proof.

We check that there is a polynomial $P_n$ such that

$$f^{(n)}(x) = \begin{cases} P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

This is true if $n = 0$. If it holds for some $n \geq 0$, then for $x \neq 0$, differentiating with the product rule gives

$f^{(n+1)}(x) = \left(-P_n'\left(\frac{1}{x}\right)\frac{1}{x^2} + \frac{2P_n\left(\frac{1}{x}\right)}{x^3}\right) e^{-\frac{1}{x^2}}$, as required. □

# Exponential limits

**Proof.**

To check that $f^{(n+1)}(0) = 0$ write

$$\lim_{x \to 0} \frac{P_n\left(\frac{1}{x}\right)}{x} e^{-\frac{1}{x^2}}.$$

Substitute $x \mapsto \frac{1}{x}$ and note that

$$\lim_{x \to \infty} x P_n(x) e^{-x^2} = 0, \qquad \lim_{x \to -\infty} x P_n(x) e^{-x^2} = 0$$

to evaluate the right and left limits at 0. $\square$

# Trig substitution

The substitution $u = \tan \frac{x}{2}$ permits integration of rational functions in sin and cos.

## Problem

*Integrate* $\int \frac{dx}{\sin x + \cos x}$.

## Solution

*Substitute* $u = \tan \frac{x}{2}$ *to obtain*

$$x = 2 \arctan u, \qquad dx = \frac{2du}{1+u^2},$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2 \tan \frac{x}{2}}{\sec^2 \frac{x}{2}} = \frac{2u}{1+u^2},$$

$$\cos x = 2 \cos^2 \frac{x}{2} - 1 = \frac{2}{\sec^2 \frac{x}{2}} - 1 = \frac{2}{1+u^2} - 1 = \frac{1-u^2}{1+u^2}.$$

# Trig substitution

## Solution

*Write* $\sin x + \cos x = \frac{2u+1-u^2}{1+u^2}$ *to write*

$$\begin{aligned}\int \frac{dx}{\sin x + \cos x} &= -2\int \frac{du}{u^2 - 2u - 1}\\ &= -2\int \frac{du}{(u-1-\sqrt{2})(u-1+\sqrt{2})}\\ &= \frac{1}{\sqrt{2}}\log\left|\frac{u-1+\sqrt{2}}{u-1-\sqrt{2}}\right| + C\\ &= \frac{1}{\sqrt{2}}\log\left|\frac{\tan\frac{x}{2}-1+\sqrt{2}}{\tan\frac{x}{2}-1-\sqrt{2}}\right| + C.\end{aligned}$$

# Improper integrals

## Definition

The infinite integral is defined by a limit, if it exists,

$$\int_a^\infty f(x)dx = \lim_{b\to+\infty} \int_a^b f(x)dx,$$

and

$$\int_{-\infty}^a f(x)dx = \lim_{b\to-\infty} \int_b^a f(x)dx.$$

Define, for any $c \in \mathbb{R}$,

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx.$$

If an infinite integral exists, it is said to be convergent.

# Examples

- $\int_0^1 x^{s-1}dx$ converges for $s > 0$ and diverges for $s \leq 0$, since

$$\int_b^1 x^{s-1}dx = \begin{cases} \frac{1-b^s}{s} & s \neq 0, \\ -\log b & s = 0 \end{cases}.$$

- $\int_0^\infty \sin x dx$ does not converge, since

$$\int_0^b \sin x dx = 1 - \cos b$$

  varies between 0 and 2 for arbitrarily large $b$.
- $\int_0^\infty e^{-x}dx = 1$ since $\int_0^a e^{-x}dx = 1 - e^{-a}$ for $a > 0$.

# Improper integrals of a second kind

## Definition

Suppose that $f$ is bounded and integrable on subintervals $[x, b]$ of $[a, b]$. If $f$ is unbounded on $[a, b]$, it's improper integral, if it exists, is

$$\int_a^b f(x)dx = \lim_{x \downarrow a} \int_x^b f(x)dx.$$

# The Gamma function

## Definition

Let $s > 0$. The Gamma function $\Gamma(s)$ is defined by the improper integral

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx.$$

# The Gamma function

- When $0 < s < 1$ the above integral is improper at both endpoints, and is defined by

$$\Gamma(s) = \int_0^1 e^{-x} x^{s-1} dx + \int_1^\infty e^{-x} x^{s-1} dx.$$

- To check that the first integral converges, note that $\int_a^1 e^{-x} x^{s-1} dx$ is continuous and decreasing as a function of $a$ and bounded above by $\frac{1}{s} = \int_0^1 x^{s-1} dx$, hence has a limit equal to its supremum at $a = 0$.

# The Gamma function

- To check that the second integral converges, note that

$$\int_1^a e^{-x}x^{s-1}dx = \left(\frac{1}{e} - a^{s-1}e^{-a}\right) + (s-1)\int_1^a e^{-x}x^{s-2}dx$$

  is an increasing function of $a$. If $0 < s \leq 1$ the integral is bounded by $\int_1^\infty e^{-x}dx = \frac{1}{e}$. Since it is increasing and bounded above, it converges to its supremum, see HW9. To check that the integral converges for all $s$, use the integration by parts formula to replace $s$ with $s-1$; apply induction.

# Properties Gamma function

## Theorem

*The $\Gamma$ function satisfies the following properties.*

1. $\Gamma(1) = 1$
2. *For* $s \geq 0$, $s\Gamma(s) = \Gamma(s+1)$. *In particular, for integers* $n \geq 1$, $\Gamma(n) = (n-1)!$.
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

## Proof.

$\Gamma(1) = \int_0^\infty e^{-x} dx = 1$. To prove the second relation, integrate by parts in

$$\int_a^b e^{-x} x^{s-1} dx = a^{s-1}e^{-a} - b^{s-1}e^{-b} + (s-1)\int_a^b e^{-x} x^{s-2} dx.$$

Taking the limit, first as $b \to \infty$, then as $a \to 0$, the evaluation terms both vanish, leaving $\Gamma(s) = (s-1)\Gamma(s-1)$. $\square$

# Properties Gamma function

**Proof.**

We give a 'proof sketch' of the last relation, since the easiest proof uses multiple integrals, which we don't treat in this course. Substitute $u = \sqrt{x}$, $2du = \frac{dx}{\sqrt{x}}$ to write

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x} \frac{dx}{\sqrt{x}}$$
$$= 2\int_0^\infty e^{-u^2} du = \int_{-\infty}^\infty e^{-u^2} du.$$

Thus $\Gamma\left(\frac{1}{2}\right)^2 = \int_{x,y\in\mathbb{R}^2} e^{-x^2-y^2} dxdy$. Note that this requires justification that the two integrals over $x$ and $y$ can be merged into 1, which we don't cover in this course. $\square$

# Properties Gamma function

## Proof.

Change to polar coordinates $(r, \theta)$ where $r^2 = x^2 + y^2$ and $\theta = \arctan(y/x)$. The function is independent of $\theta$, hence is constant on concentric circles about the origin. The area of the circular rim between $r$ and $r + \delta$ is $2\pi r\delta + O(\delta^2)$, and hence the integral has value

$$\Gamma\left(\frac{1}{2}\right)^2 = 2\pi \int_0^\infty e^{-r^2} r dr = \pi \int_0^\infty e^{-x} dx = \pi.$$

$\square$

# The Gaussian

The function $G(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ is the density function of a 'Gaussian' or 'normal' distribution (the 'bell curve'). It is ubiquitous in statistics and mathematical analysis. By the previous theorem,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = 1.$$

The numbers $M(n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} dx$ are called the moments of the Gaussian. For odd $n$ these vanish since the integrand is odd (pair $x$ and $-x$). Set $m(n) = M(2n)$ for the even moments.

# Moments of the Gaussian

The even moments of the Gaussian are given by

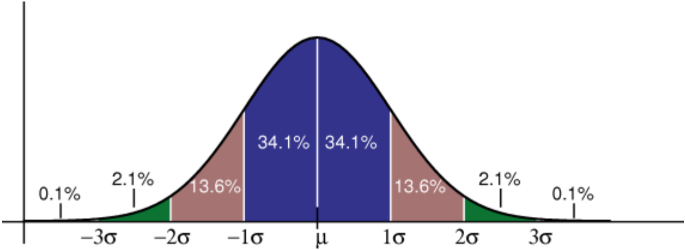
$$m(n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-\frac{x^2}{2}} dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} x^{2n} e^{\frac{-x^2}{2}} dx.$$

Substitute $u = \frac{x^2}{2}$, $du = xdx$ to find

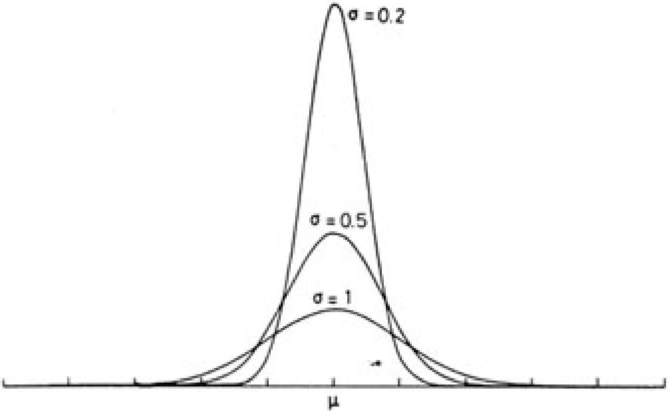
$$\begin{aligned} m(n) &= \frac{2^n}{\sqrt{\pi}} \int_0^{\infty} u^{n-\frac{1}{2}} e^{-u} du \\ &= 2^n \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \\ &= 2^n \frac{1}{2}\frac{3}{2} \ldots \left(n - \frac{1}{2}\right) = 1 \cdot 3 \cdot \ldots \cdot (2n-1). \end{aligned}$$

# Gauss

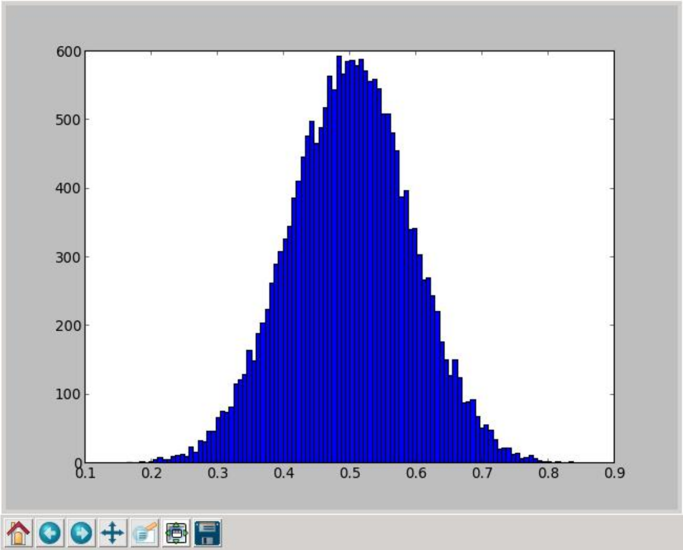
# Gaussian



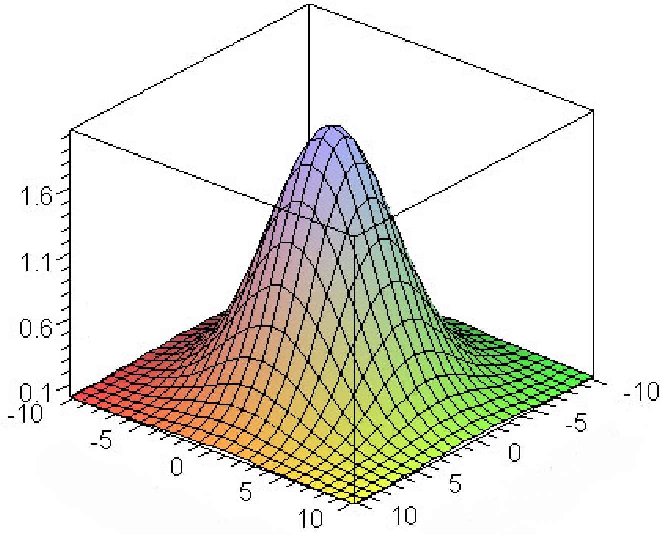
# Change of variance



# Fitting the bell curve



# Multi-dimensional Gaussian



# The sup distance on continuous functions

## Definition

Let $I$ be an interval, and let $f$ be a bounded function on $I$. The sup norm of $f$ is

$$\|f\|_\infty = \sup_{x \in I} |f(x)|.$$

A sequence of bounded functions $\{f_n\}_{n=0}^\infty$ *converges uniformly* to a bounded function $f$ if

$$\lim_{n \to \infty} \|f - f_n\|_\infty = 0.$$

# The sup distance on continuous functions

**Theorem**

*Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of continuous functions on $[a, b]$ converging uniformly to $f$. Then $f$ is continuous.*

Remark: we say that the set of continuous functions on $[a, b]$ is *closed* under uniform convergence.

**Proof.**

Given $\epsilon > 0$ choose $N$ sufficiently large so that $n > N$ implies $\|f - f_n\|_\infty < \frac{\epsilon}{3}$. Pick any $n > N$, and let $\delta > 0$ be such that $|y - x| < \delta$ implies $|f_n(y) - f_n(x)| < \frac{\epsilon}{3}$. Then, by the triangle inequality,

$$|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)| < \epsilon.$$

This proves that $f$ is uniformly continuous on $[a, b]$. $\square$

# Pointwise convergence

## Definition

Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of functions on an interval $I$, and let $f$ be a function on $I$. The sequence $\{f_n\}_{n=0}^{\infty}$ *converges pointwise* to $f$ if, for each $x \in I$,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

# Example of pointwise convergence

Let $\{f_n(x) = x^n\}$ be functions on $[0, 1]$, and let $f(x) = 0$ if $0 \leq x < 1$ and $f(1) = 1$, otherwise.

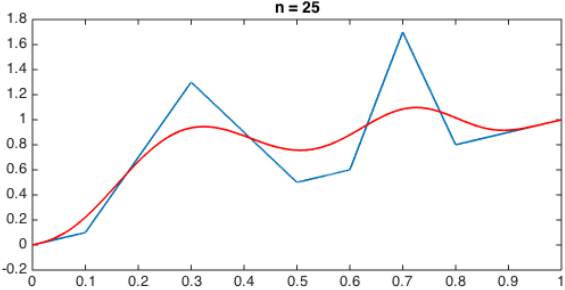
- Then $f_n$ converges to $f$ pointwise, since for all $n$, $1^n = 1$, $0^n = 0$, while for all $0 < x < 1$, $x^n = \exp(n \log x)$ tends to 0 as $n \to \infty$, since $\log x < 0$.
- $f_n$ does not tend to $f$ uniformly, and in fact, $\|f_n - f\|_\infty = 1$ for all $n$, since $f_n(x) \to 1$ as $x \to 1$ for each $n$.

# The Weierstrass Approximation Theorem

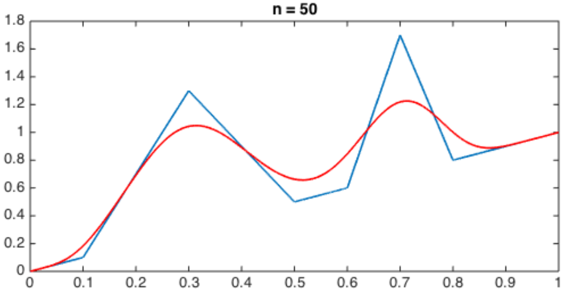
**Theorem (Weierstrass Approximation Theorem)**

*Let $f$ be a continuous function on $[-1, 1]$. There exists a sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ converging uniformly to $f$ on $[-1, 1]$.*

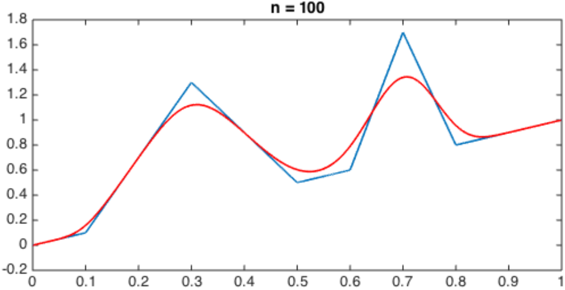
# The Weierstrass Approximation Theorem Illustration



# The Weierstrass Approximation Theorem Illustration



# The Weierstrass Approximation Theorem Illustration



# The Weierstrass Approximation Theorem

## Proof.

Let $Q_n(x) = c_n(1-x^2)^n$ where $c_n$ is a constant such that $\int_{-1}^1 Q_n(x)dx = 1$. These polynomials satisfy

1. $Q_n \geq 0$
2. For each $\delta > 0$, $\lim_{n\to\infty} \int_{\delta<|x|\leq 1} Q_n(x)dx = 0$.

To prove the second item, note that $\frac{\int_{\delta<|x|\leq 1} Q_n(x)dx}{\int_{\delta<|x|\leq 1} Q_{n-1}(x)dx} \leq (1-\delta^2)\frac{c_n}{c_{n-1}}$, while $\frac{\int_{|x|<\delta/2} Q_n(x)dx}{\int_{|x|<\delta/2} Q_{n-1}(x)dx} \geq \frac{c_n}{c_{n-1}}(1-\delta^2/4)$, so that

$$\frac{\int_{\delta<|x|\leq 1} Q_n(x)dx}{\int_{|x|<\delta/2} Q_n(x)dx}, \qquad n = 1, 2, ...$$

decreases by a factor of $\frac{1-\delta^2}{1-\delta^2/4} < 1$ at each step $n = 1, 2, ...$, so tends to 0. Since each denominator is bounded by 1, the numerator tends to 0.

# The Weierstrass Approximation Theorem

## Proof.

- Extend $f$ to a function continuous on $\mathbb{R}$ by setting $f(x) = f(-1)$ for $x < -1$ and $f(x) = f(1)$ for $x > 1$.
- Let $|f| \leq M$.
- Given $\epsilon > 0$, since $f$ is uniformly continuous there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{2}$.
- Choose $N$ such that $n > N$ implies $\int_{\delta < |x| \leq 1} Q_n(x) < \frac{\epsilon}{4M}$.
- Define, for $x \in [-1, 1]$,
  $P_n(x) = \int_{-1}^{1} Q_n(t) f(x - t) dt = \int_{x-1}^{x+1} f(t) Q_n(x - t) dt$.
- Note that this last expression is a polynomial in $x$ after the $t$ variable is integrated away.

# The Weierstrass Approximation Theorem

Proof.

- Recall $P_n(x) = \int_{-1}^1 Q_n(t) f(x-t)dt$, and rewrite this integral as

$$P_n(x) = f(x) + \int_{-1}^1 Q_n(t)(f(x-t) - f(x))dt.$$

- Thus

$$\begin{aligned}
&|P_n(x) - f(x)| \\
&\le \int_{-\delta}^{\delta} Q_n(t)|f(x-t) - f(x)|dt + \int_{\delta < |t| \le 1} Q_n(t)|f(x-t) - f(x)|dt \\
&< \frac{\epsilon}{2}\int_{-\delta}^{\delta} Q_n(t)dt + 2M\int_{\delta<|t|\le 1} Q_n(t)dt \le \frac{\epsilon}{2} + \frac{2M\epsilon}{4M} = \epsilon.
\end{aligned}$$

$\square$

# A moment theorem

## Theorem

*If f is continuous on* $[0,1]$ *and if for* $n = 0, 1, 2, \ldots,$

$$\int_0^1 f(x)x^n dx = 0$$

*then* $f(x) = 0$ *on* $[0,1]$.

# A moment theorem

The proof uses the following lemma.

## Lemma

*Let $f$ on $[0,1]$ be continuous. If $\int_0^1 f(x)^2 dx = 0$ then $f(x) = 0$ for all $x \in [0,1]$.*

## Proof.

Suppose for contradiction that there is $x \in (0,1)$ with $f(x) \neq 0$. Choose $\delta > 0$ such that $|y - x| < \delta$ implies $y \in [0,1]$ and $|f(y) - f(x)| < \frac{|f(x)|}{2}$. In particular, $|f(y)| > \frac{|f(x)|}{2}$. It follows that

$$\int_0^1 f(x)^2 dx \geq \int_{x-\delta}^{x+\delta} f(y)^2 dy \geq 2\delta \frac{|f(x)|^2}{4} > 0.$$

$\square$

# A moment theorem

## Proof of moment theorem.

Let $|f| \leq M$ on $[0,1]$. Given $\epsilon > 0$, apply the Weierstrass Approximation Theorem to choose a polynomial $P$ such that $\sup_{x\in[0,1]} |f(x) - P(x)| < \epsilon$. Write

$$\begin{aligned}\int_0^1 f(x)^2 dx &= \int_0^1 f(x)(f(x) - P(x))dx + \int_0^1 f(x)P(x)dx \\ &\leq M\epsilon + 0.\end{aligned}$$

Letting $\epsilon \downarrow 0$ proves $\int_0^1 f(x)^2 dx = 0$, so $f(x) = 0$ for all $x \in [0,1]$. □