Math 141: Lecture 13

Taylor polynomials and indeterminants

Bob Hough

October 19, 2016

The Taylor Polynomial of a function

Definition

Let f be n-times differentiable at a point a. The degree n Taylor polynomial of f at a is

$$T_n f(x;a) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j.$$

The degree n Taylor polynomial satisfies, for each $0 \leq j \leq n$,

$$\frac{d^j}{dx^j} T_n f(x;a)\Big|_{x=a} = f^{(j)}(a).$$

Algebra of Taylor Polynomials

Theorem

Let f and g have n derivatives at the point a. The following functions have n derivatives at a, with Taylor polynomials given.

1. *If c_1 and c_2 are constants, $c_1 f + c_2 g$ has* $T_n(c_1 f + c_2 g) = c_1 T_n(f) + c_2 T_n(g)$.
2. *fg has $T_n(fg) = T_n(T_n(f)T_n(g))$.*
3. *If $g(a) \neq 0$, $T_n(f/g) = T_n(T_n(f)/T_n(g))$.*

Proof.

Formulas for the higher derivatives of sums, products and ratios of functions can be developed through the rules for first derivatives, and the chain rule. In each case, the formula for the nth derivative involves only the first n derivatives of the original functions. Hence the answer depends only on the degree n Taylor polynomials themselves. $\square$

Algebra of Taylor Polynomials

1. The degree n Taylor expansion of $\frac{1}{1-x}$ about 0 is found by solving

$$(c_0 + c_1 x + c_2 x^2 + ... + c_n x^n)(1 - x) = 1.$$

   Matching the constant coefficient, $c_0 = 1$. Equating the remaining coefficients gives $c_j = c_{j-1}$ for $j = 1, 2, 3, ...$, so all coefficients are 1 as expected.

2. The degree n Taylor expansion of $\frac{1}{(1-x)^2}$ is given by taking the first n terms in the expansion

$$a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ... = (1 + x + ... + x^n)(1 + x + ... + x^n).$$

   One finds $a_0 = 1$, $a_1 = 2$, ..., $a_n = n + 1$.

Calculus of Taylor Polynomials

Theorem

The Taylor polynomial operator satisfies the following properties.

1. *Substitution: Let c be a non-zero constant and let $g(x) = f(cx)$. Then $T_n g(x; a) = T_n f(cx; ca)$.*
2. $T_n(f)' = T_{n-1}(f')$.
3. *If $g(x) = \int_a^x f(t)dt$, then $T_{n+1}g(x; a) = \int_a^x T_n f(t)dt$.*

Proof.

For the first item, use the chain rule with $g(x) = f(cx)$ to obtain

$$g'(x) = cf'(cx),\ g''(x) = c^2 f''(cx),\ \ldots,\ g^{(k)}(x) = c^k f^{(k)}(cx),$$

$$T_n g(x; a) = \sum_{k=0}^{n} \frac{g^{(k)}(a)}{k!}(x - a)^k = \sum_{k=0}^{n} \frac{f^{(k)}(ca)}{k!}(cx - ca)^k = T_n f(cx; ca).$$

□

Calculus of Taylor Polynomials

Proof.

To prove the second property, note that both $(T_n f)'$ and $T_{n-1}(f')$ are degree $n-1$ polynomials. Since all $n-1$ derivatives match, the polynomials are equal. The same argument applies to show that

$$T_{n+1}g(x;a) = \int_a^x T_n f(t;a)dt.$$

$\square$

Some well-known Taylor Polynomials

1. Differentiating the formula $T_n\left(\frac{1}{1-x}\right)$ gives a new proof of the formula above

$$T_{n-1}\left(\frac{1}{(1-x)^2}\right) = 1 + 2x + 3x^2 + ... + nx^{n-1}.$$

2. Integrating instead

$$T_{n+1}\left[-\log(1-x)\right] = x + \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^{n+1}}{n+1}.$$

A Taylor Theorem

Theorem

Let P_n be a polynomial of degree $n \geq 1$. Let f and g be two functions with derivatives of order n at 0 and assume that

$$f(x) = P_n(x) + x^n g(x)$$

where $g(x) \to 0$ as $x \to 0$. Then P_n is the Taylor polynomial generated by f at 0.

Proof.

Let $h(x) = f(x) - P_n(x) = x^n g(x)$. Differentiating the product $x^n g$ n times shows that h and its first n derivatives are 0 at 0. □

Some well-known Taylor Polynomials

1. For any $n \geq 1$,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots + (-1)^n x^{2n} - (-1)^n \frac{x^{2(n+1)}}{1+x^2}.$$

   Hence $T_{2n}\left(\frac{1}{1+x^2}\right) = 1 - x^2 + x^4 - \dots + (-1)^n x^{2n}$.

2. Integrating the previous formula gives

$$T_{2n+1}(\arctan x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1}.$$

Some well-known Taylor Polynomials

1. The equation $\frac{d}{dx}e^x = e^x$ gives
$$T_n\left(e^x\right) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}.$$
2. Substituting $-x$ for x gives
$$T_n\left(e^{-x}\right) = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \ldots + (-1)^n\frac{x^n}{n!}.$$

Some well-known Taylor Polynomials

1. The hyperbolic cosine is defined by $\cosh x = \frac{e^x+e^{-x}}{2}$.

$$T_{2n}\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ... + \frac{x^{2n}}{(2n)!}.$$

2. The hyperbolic sine is defined by $\sinh x = \frac{e^x-e^{-x}}{2}$.

$$T_{2n}\sinh(x) = x + \frac{x^3}{3!} + ... + \frac{x^{2n-1}}{(2n-1)!}.$$

Some well-known Taylor Polynomials

1. By the formulas, $\sin 0 = 0$, $\cos 0 = 1$, $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$,

$$T_{2n}\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

2. Similarly

$$T_{2n}\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n\frac{x^{2n}}{(2n)!}.$$

Some well-known Taylor Polynomials

1. Use $\cos x \sec x = 1$ to solve for

$$\sec x = c_0 + c_2 x^2 + c_4 x^4 + \ldots,$$

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right)\left(c_0 + c_2 x^2 + c_4 x^4 + c_6 x^6 + \ldots\right) = 1,$$

so $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots$.

2. Use $\tan x = \sin x \sec x$ to find

$$T_5(\tan x) = T_5\left(\left(1 + \frac{x^2}{2} + \frac{5x^4}{24}\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\right)$$
$$= x + \frac{x^3}{3} + \frac{2x^5}{15}.$$

Taylor's formula with remainder

Let f have n derivatives at a. Write

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + E_n(x).$$

$E_n(x)$ is called the remainder in Taylor's approximation.

Taylor's formula with remainder

Theorem

Assume that f has a continuous second derivative in a neighborhood of a. Then

$$f(x) = f(a) + (x-a)f'(a) + \int_a^x (x-t)f''(t)dt.$$

Proof.

By the FTC, $f(x) - f(a) = \int_a^x f'(t)dt = (x-a)f'(a) + \int_a^x f'(t) - f'(a)dt$. In the integral, integrate by parts using $u = (f'(t) - f'(a))$ and $dv = d(t-x)$. Thus

$$f(x) = f(a) + (x-a)f'(a) + \int_a^x (x-t)f''(t)dt.$$

$\square$

Taylor's formula with remainder

Theorem

Let f have a continuous derivative of order $n+1$ in an interval containing a. For every x in this interval,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt.$$

Taylor's formula with remainder

Proof.

The proof is by induction. The base case was proved in the previous theorem, so suppose this holds for some $n \geq 1$. Notice that $\frac{(x-a)^{n+1}}{(n+1)!} = \frac{1}{n!}\int_a^x (x-t)^n dt$. By the inductive assumption

$$f(x) = \sum_{k=0}^{n+1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x (x-t)^n [f^{(n+1)}(t) - f^{(n+1)}(a)] dt.$$

In the integral, integrate by parts with $u = f^{(n+1)}(t) - f^{(n+1)}(a)$, $dv = (x-t)^n dt$ to find

$$f(x) = \sum_{k=0}^{n+1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{(n+1)!}\int_a^x (x-t)^{n+1} f^{(n+2)}(t) dt.$$

$\square$

The irrationality of e

Theorem

Euler's constant e is irrational.

Proof.

Suppose $e = \frac{p}{q}$ with $q > 2$, is rational. By Taylor expanding e^x about 0, write

$$e = e^1 = \sum_{n=0}^{q} \frac{1}{n!} + \frac{1}{q!} \int_0^1 (1-x)^q e^x dx.$$

Hence $q!e$ is an integer, equal to

$$\sum_{n=0}^{q} \frac{q!}{n!} + \int_0^1 (1-x)^q e^x dx.$$

The sum is an integer. The integral is positive, and bounded by $e \int_0^1 (1-x)^q dx = \frac{e}{q+1} < 1$, a contradiction. $\square$

Other forms of the remainder in Taylor's formula

Theorem

If the $(n+1)$st derivative of f satisfies $m \le f^{(n+1)}(t) \le M$ for all t in an interval about a, then

$$m\frac{(x-a)^{n+1}}{(n+1)!} \le E_n(x) \le M\frac{(x-a)^{n+1}}{(n+1)!} \qquad \text{if } x > a,$$

$$m\frac{(a-x)^{n+1}}{(n+1)!} \le (-1)^{n+1}E_n(x) \le M\frac{(a-x)^{n+1}}{(n+1)!} \qquad \text{if } x < a.$$

Proof.

If $x > a$, write $E_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt$ and insert the lower and upper bounds for $f^{(n+1)}(t)$ to conclude. If $x < a$,

$$(-1)^{n+1}E_n(x) = \frac{1}{n!}\int_x^a (t-x)^n f^{(n+1)}(t)dt,$$

and argue as before. □

Little o and Big O

Definition

Assume $g(x) \neq 0$ for all $x \neq a$ in an interval containing a. The notation

$$f(x) = o(g(x)) \qquad \text{as } x \to a$$

means $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$.

Definition

Assume $g(x) \neq 0$ for all $x \neq a$ in an interval containing a. The notation

$$f(x) = O(g(x)) \qquad \text{as } x \to a$$

if there is a constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all $x \neq a$ in a neighborhood of a.

Little o and Big O examples

1. $f(x) = o(1)$ as $x \to a$ means $f(x) \to 0$ as $x \to a$.
2. $f(x) = o(x)$ as $x \to 0$ means $\lim_{x\to 0} \frac{f(x)}{x} = 0$.
3. $\sin x = x + O(x^3)$ as $x \to 0$. This follows from writing $\sin x = x - \frac{1}{2}\int_0^x (x-t)^2 \cos t dt$. The bound $O(x^3)$ follows by bounding $|\cos t| \leq 1$.

Algebra of little o and Big O

Theorem

As $x \to a$, the following hold:

1. *If $f, g = o(h)$ then $f \pm g = o(h)$. If $f, g = O(h)$ then $f \pm g = O(h)$.*
2. *If $c \neq 0$, if $f = o(h)$ then $cf = o(h)$. If $f = O(g)$ then $cf = O(g)$.*
3. *If $f(x) \neq 0$ in a neighborhood of a and if $g = o(h)$ then $fg = o(fh)$. If $g = O(h)$ then $fg = O(fh)$.*

Proof.

If $\lim_{x\to a} \frac{f(x)}{h(x)} = 0$ and $\lim_{x\to a} \frac{g(x)}{h(x)} = 0$ then, by the linearity of limits $\lim_{x\to a} \frac{f(x)\pm g(x)}{h(x)} = 0$ and $\lim_{x\to a} \frac{cf(x)}{h(x)} = 0$. This proves the the first two o claims. If $|f| \leq C_1|h|$ and $|g| \leq C_2|h|$ then by the triangle inequality $|f \pm g| \leq (C_1 + C_2)|h|$ and $|cf| \leq |c|C_1|h|$. This proves the first two big O claims. Both of the last claims follow by canceling factors of $|f|$. $\square$

Algebra of little o and Big O

Theorem

As $x \to a$, the following hold:

1. *If $f = o(g)$ and $g = o(h)$ then $f = o(h)$. If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.*
2. *If $g = o(1)$ then $\frac{1}{1+g} = 1 - g(x) + o(g(x))$.*

Proof.

1. If $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{g(x)}{h(x)} = 0$, then $\lim_{x\to a} \frac{f(x)}{h(x)} = \lim_{x\to a} \frac{f(x)}{g(x)} \frac{g(x)}{h(x)} = 0$.
2. If $|f(x)| \leq C_1|g(x)|$ and $|g(x)| \leq C_2|h(x)|$ then $|f(x)| \leq C_1C_2|h(x)|$.
3. To prove the last claim $\frac{1}{1+g(x)} = 1 - g(x) + g(x)\frac{g(x)}{1+g(x)}$. Observe $\lim_{x\to a} \frac{g(x)}{1+g(x)} = 0$.

$\square$

Little o and Big O examples

Problem

Prove that $(1+x)^{\frac{1}{x}} = e\left(1 - \frac{x}{2} + \frac{11x^2}{24} + o(x^2)\right)$ *as* $x \to 0$.

Solution

Write $\frac{\log(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} + o(x^2)$ *so that*

$$(1+x)^{\frac{1}{x}} = \exp(1 - x/2 + x^2/3 + o(x^2)) = e \cdot e^u.$$

with $u = -x/2 + x^2/3 + o(x^2)$. *As* $x \to 0$, $u = O(x)$ *tends to 0.* *Furthermore, as* $u \to 0$, $e^u = 1 + u + u^2/2 + o(u^2)$ *and so*

$$e^u = 1 - \frac{x}{2} + \frac{x^2}{3} + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} + o(x^2)\right)^2 + o(x^2) = 1 - \frac{x}{2} + \frac{11x^2}{24} + o(x^2).$$

Examples of limits

Problem

Prove that $\lim_{x \to 0} \frac{\log(1+ax)}{x} = a$.

Solution

Write $\log(1 + ax) = ax + o(x)$ *to conclude.*

Examples of limits

Problem

Calculate for $b \neq 0$, $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$.

Solution

Write $\sin ax = ax + o(x)$, $\sin bx = bx + o(x)$. *Thus* $\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}$.

Examples of limits

Problem

Calculate $\lim_{x\to 0} \frac{\sin x}{\arctan x}$.

Solution

From $\sin x = x + o(x)$ *and* $\arctan x = x + o(x)$*, deduce that the limit is 1.*

Examples of limits

Problem

Let $a, b > 0$ *with* $b \neq 1$. *Calculate* $\lim_{x\to 0} \frac{a^x-1}{b^x-1}$.

Solution

Write $a^x = \exp(x \log a)$ *and* $b^x = \exp(x \log b)$. *Hence* $a^x = 1 + x\log a + o(x)$ *and* $b^x = 1 + x \log b + o(x)$. *It follows that the limit is*

$$\frac{\log a}{\log b}.$$

l'Hôpital's Rule

Theorem

Assume f and g have derivatives $f'(x)$ and $g'(x)$ at each point x of an open interval (a, b). Assume

$$\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = 0$$

and that $g'(x) \neq 0$ for all $x \in (a, b)$. If the limit $\lim_{x \to a+} \frac{f'(x)}{g'(x)}$ exists then

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f'(x)}{g'(x)}.$$

l'Hôpital's Rule

Proof.

Define $F(x) = f(x)$ if $x \neq a$ and $F(a) = 0$, $G(x) = g(x)$ if $x \neq a$, $G(a) = 0$. Thus F and G are continuous on $[a, x]$ and differentiable on (a, x) for $a < x < b$. By Cauchy's Mean Value Theorem, there is $a < c < x$ such that

$$[F(x) - F(a)]G'(c) = [G(x) - G(a)]F'(c),$$

which simplifies to $\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$. Letting $x \to a+$,

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f'(x)}{g'(x)}.$$

□

Examples of l'Hôpital's Rule

Problem

Evaluate

$$\lim_{x \to 2} \frac{3x^2 + 2x - 16}{x^2 - x - 2}.$$

Solution

Numerator and denominator vanish. By l'Hôpital, the limit equals

$$\lim_{x \to 2} \frac{6x + 2}{2x - 1} = \frac{14}{3}.$$

Examples of l'Hôpital's Rule

Problem

Let $a > 0$. Evaluate

$$\lim_{x \to a+} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2 - a^2}}$$

Solution

Both numerator and denominator vanish as $x \to a$. By l'Hôpital, the limit is

$$\lim_{x \to a+} \frac{\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x-a}}}{\frac{x}{\sqrt{x^2-a^2}}} = \lim_{x \to a+} \frac{\sqrt{x^2 - a^2}}{2x^{\frac{3}{2}}} + \frac{\sqrt{x+a}}{2x} = \frac{1}{\sqrt{2a}}.$$

Examples of l'Hôpital's Rule

Problem

Evaluate $\lim_{x\to 0} \frac{\log\cos ax}{\log\cos bx}$.

Solution

Since numerator and denominator tend to 0 we can apply l'Hôpital to find that the limit is

$$\lim_{x\to 0} \frac{a\sin ax\cos bx}{b\sin bx\cos ax}.$$

The cosine terms tend to 1, so can be removed. Write $\sin ax = ax + o(x)$ *and* $\sin bx = bx + o(x)$ *to conclude that the limit is* $\frac{a^2}{b^2}$.

l'Hôpital's Rule for infinite limits

Theorem

Let f and g be differentiable on (a, b) and satisfy $\lim_{x\to a+} f(x) = \lim_{x\to a+} g(x) = \infty$. If $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L$ exists then $\lim_{x\to a+} \frac{f(x)}{g(x)} = L$.

Limits involving exponentials and logs

Theorem

If $a > 0$ and $b > 0$, we have

$$\lim_{x\to\infty} \frac{(\log x)^b}{x^a} = 0,$$

and

$$\lim_{x\to\infty} \frac{x^b}{e^{ax}} = 0.$$

Proof.

For $r > 0$, the limit $\lim_{x\to\infty} \frac{\log x}{x^r} = \lim_{x\to\infty} \frac{\frac{1}{x}}{rx^{r-1}} = \lim_{x\to\infty} \frac{1}{rx^r} = 0$. The first limit follows since the function x^b is continuous from the right at 0. For $r > 0$ the limit $\lim_{x\to\infty} \frac{x}{e^{rx}} = 0$ follows from l'Hôpital, and the limit in general follows since x^b is continuous from the right at 0. $\square$