MATH 141, FALL 2016 PRACTICE MIDTERM 2 SOLUTIONS

NOVEMBER 2

Solve 4 of 6 problems. You may quote any result stated during lecture, so long as you represent the result accurately.

Problem 1.

a. (2 points) State carefully the Chain Rule of differential calculus.

b. (3 points) For integer $n \geq 1$, define the n-times iterated logarithm by $\log_{(1)} x = \log x$, and, for $n \geq 1$, and x such that $\log_{(n)}(x) > 0$, $\log_{(n+1)} x = \log(\log_{(n)} x)$. Derive a formula for $\frac{d}{dx} \log_{(n)} x$.

Solution.

a. Let f be differentiable at a and g be differentiable at $f(a)$. Then $g \circ f$ is differentiable at a, and $\frac{d}{dx} g \circ f \big|_{x=a} = g'(f(a))f'(a)$.

b. The formula is $\frac{d}{dx} \log_{(n)} x = \frac{1}{x} \prod_{1 \leq j < n} \frac{1}{\log_{(j)} x}$.

We prove the formula by induction.

Base case ($n = 1$): This is just the usual derivative formula for the logarithm, $\frac{d}{dx} \log x = \frac{1}{x}$.

Inductive step: For $n \geq 1$, by the chain rule,

$$\begin{aligned} \frac{d}{dx} \log_{(n+1)} x &= \frac{1}{\log_{(n)} x} \frac{d}{dx} \log_{(n)}(x) \\ &= \frac{1}{\log_{(n)} x} \frac{1}{x} \prod_{1 \leq j < n} \frac{1}{\log_{(j)} x} \\ &= \frac{1}{x} \prod_{1 \leq j < n+1} \frac{1}{\log_{(j)} x}. \end{aligned}$$

Problem 2.

a. (2 points) State carefully the Mean Value Theorem for a function on an interval $[a, b]$.

b. (3 points) Prove that if f is n-times differentiable on (a, b) and $f(x) = 0$ for $n + 1$ different x in (a, b), then $f^{(n)}(x) = 0$ for some $x \in (a, b)$.

Solution.

a. Let $a < b$. Let f be continuous on $[a, b]$ and differentiable on (a, b). There exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

b. The proof is by induction.

Base case ($n = 1$): Let $f(x_1) = f(x_2) = 0$ for some $a < x_1 < x_2 < b$. By the Mean Value Theorem applied to f on $[x_1, x_2]$, there is a zero of $f'(x)$ with $x_1 < x < x_2$.

Inductive step: Suppose the statement of the result holds for some $n \geq 1$ and let f be $n + 1$-times differentiable on (a, b) with $n + 2$ zeros at $\{x_i\}_{i=1}^{n+2}$, with $a < x_1 < x_2 < ... < x_{n+2} < b$. Applying the base case in each interval $[x_i, x_{i+1}]$ for $1 \leq i \leq n + 1$, find that f' has zero $y_i \in (x_i, x_{i+1})$, and hence f' is n times differentiable with $n+1$ distinct zeros at $\{y_i\}_{i=1}^{n+1}$, $a < y_1 < y_2 < ... < y_{n+1} < b$. By the inductive assumption, $f^{(n+1)} = (f')^{(n)}$ has a zero in (a, b).

Problem 3. Use integration by parts to derive the formula for $m, n \geq 1$,

$$\int \frac{\sin^{n+1} x}{\cos^{m+1} x} dx = \frac{1}{m} \frac{\sin^n x}{\cos^m x} - \frac{n}{m} \int \frac{\sin^{n-1} x}{\cos^{m-1} x} dx.$$

Apply the formula to integrate $\tan^2 x$ and $\tan^4 x$.

Solution. Let $du = \frac{\sin x}{\cos^{m+1} x} dx$ and $v = \sin^n x$, so that $u = \frac{1}{m} \frac{1}{\cos^m x}$ and $dv = n \sin^{n-1} x \cos x dx$. Integrating by parts

$$\int \frac{\sin^{n+1} x}{\cos^{m+1} x} dx = \frac{1}{m} \frac{\sin^n x}{\cos^m x} - \frac{n}{m} \int \frac{\sin^{n-1} x}{\cos^{m-1} x} dx$$

as required.

It follows from the $m = n = 1$ case that

$$\int \tan^2 x dx = \tan x - \int dx = \tan x - x + C.$$

It follows from the $m = n = 3$ case that

$$\begin{aligned} \int \tan^4 x dx &= \frac{1}{3} \tan^3 x - \int \tan^2 x dx \\ &= \frac{1}{3} \tan^3 x - \tan x + x - C. \end{aligned}$$

Problem 4. Evaluate
$$\lim_{x\to\infty} xe^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{t^2}{2}} dt.$$

Solution. The substitution $u = \frac{t^2}{2}$, $du = tdt$ shows that
$$\int_1^x e^{-\frac{t^2}{2}} dt = \int_{\frac{1}{2}}^{\frac{x^2}{2}} e^{-u} \frac{du}{\sqrt{2u}}.$$

It was checked in lecture that $\lim_{x\to\infty} \int_{\frac{1}{2}}^{\frac{x^2}{2}} e^{-u} \frac{du}{\sqrt{2u}}$ exists. Thus
$$\lim_{x\to\infty} \int_x^\infty e^{-\frac{t^2}{2}} dt = \lim_{x\to\infty} \left[\int_1^\infty e^{-\frac{t^2}{2}} dt - \int_1^x e^{-\frac{t^2}{2}} dt\right] = 0.$$
Apply l'Hôpital's rule to determine
$$\lim_{x\to\infty} xe^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{t^2}{2}} dt = \lim_{x\to\infty} \frac{\int_1^\infty e^{-\frac{t^2}{2}} dt - \int_1^x e^{-\frac{t^2}{2}} dt}{\frac{1}{x} e^{-\frac{x^2}{2}}}$$
$$= \lim_{x\to\infty} \frac{-e^{-\frac{x^2}{2}}}{-\frac{1}{x^2} e^{-\frac{x^2}{2}} - e^{-\frac{x^2}{2}}} = \lim_{x\to\infty} \frac{1}{1 + \frac{1}{x^2}} = 1.$$

Note that this version of l'Hôpital's rule differs from the version proven in class, since we allow $x \to \infty$. The two versions are equivalent by substituting $y = \frac{1}{x}$ and letting $y \downarrow 0$, since
$$\frac{\frac{d}{dy} f\left(\frac{1}{y}\right)}{\frac{d}{dy} g\left(\frac{1}{y}\right)} = \frac{f'\left(\frac{1}{y}\right)\left(\frac{-1}{y^2}\right)}{g'\left(\frac{1}{y}\right)\left(\frac{-1}{y^2}\right)} = \frac{f'\left(\frac{1}{y}\right)}{g'\left(\frac{1}{y}\right)}.$$

Problem 5. Prove the following *Integral Cauchy-Schwarz Inequality.* Let f and g be continuous functions on $[a, b]$. Then

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b f(x)^2dx \int_a^b g(x)^2dx,$$

with equality if and only if $f = cg$ or $g = cf$ for some $c \in \mathbb{R}$.

Solution. Define $D(x, y)$ on $[a, b] \times [a, b]$ by

$$\begin{aligned} D(x,y) &= \frac{1}{2}\left(f(x)g(y) - f(y)g(x)\right)^2 \\ &= \frac{1}{2}\left(f(x)^2g(y)^2 + f(y)^2g(x)^2\right) - f(x)f(y)g(x)g(y) \geq 0. \end{aligned}$$

For each fixed y, $D(x, y)$ is continuous as a function of x, and hence integrable. Also, as the linear combination of continuous functions, $D_1(y) = \int_a^b D(x, y)dx$ is continuous as a function of y. If there is a point $(x, y) \in [a, b] \times [a, b]$ for which $D(x, y) > 0$, then $D_1(y) > 0$. Hence

$$\int_a^b \left(\int_a^b D(x,y)dx\right) dy \geq 0$$

with equality if and only if $D(x, y) = 0$ for all x, y.

Integrate in x first, treating y as a constant, to find

$$\begin{aligned} 0 \leq \int_a^b \left(\int_a^b D(x,y)dx\right) dy =& \frac{1}{2}\int_a^b g(y)^2 \left(\int_a^b f(x)^2dx\right) dy \\ &+ \frac{1}{2}\int_a^b f(y)^2 \left(\int_a^b g(x)^2dx\right) dy \\ &- \int_a^b f(y)g(y) \left(\int_a^b f(x)g(x)dx\right) dy \\ =& \int_a^b f(x)^2dx \int_a^b g(x)^2dx - \left(\int_a^b f(x)g(x)dx\right)^2. \end{aligned}$$

In case equality holds, if $f \equiv 0$ then the condition is met with $f = 0 \cdot g$. Otherwise, pick x with $f(x) \neq 0$ to check that $g(y) = \frac{g(x)}{f(x)} f(y)$ for all y.

Problem 6. A right triangle with hypotenuse of length a is rotated about one of its legs to generate a right circular cone. Find the greatest possible volume of such a cone.

Solution. Let the height of the cone be h and the radius of the base be r, so that $h^2 + r^2 = a^2$. The volume of the cone is

$$V(h) = \frac{\pi}{3} r^2 h = \frac{\pi}{3}(a^2 - h^2)h.$$

Thus we must maximize $V(h)$ subject to $0 \leq h \leq a$. At the endpoints $V(h) = 0$. Calculate

$$V'(h) = \frac{\pi}{3}(a^2 - 3h^2).$$

Thus $V(h)$ has a single critical point on the interval $(0, a)$ at $h = \frac{a}{\sqrt{3}}$. This is the only candidate for the maximum, and its value is

$$V_{\max} = \frac{2\pi a^3}{9\sqrt{3}}.$$