MATH 141, FALL 2016 MIDTERM 2 SOLUTIONS

NOVEMBER 2

Solve 4 of 6 problems. You may quote results stated during lecture, so long as you represent the result accurately.

Problem 1.

a. (2 points) State carefully the definition of a function differentiable at a point a.

b. (3 points) Let $\alpha > 1$. If $|f(x)| \le |x|^\alpha$, prove that f is differentiable at 0. Let $0 < \beta < 1$. Prove that if $|f(x)| \ge |x|^\beta$ and $f(0) = 0$, then f is not differentiable at 0.

Solution.

a. f is differentiable at a if the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists.

b. If $|f(x)| \le |x|^\alpha$ with $\alpha > 1$, then $|f(0)| = 0$ so $f(0) = 0$. It follows that for $x \neq 0$,

$$\left|\frac{f(x) - f(0)}{x - 0}\right| = \frac{|f(x)|}{|x|} \le |x|^{\alpha-1}.$$

Since $|x|^{\alpha-1} \to 0$ as $x \to 0$, it follows that $\lim_{x\to 0} \frac{f(x)}{x} = 0$.

If instead for $x \neq 0$, $|f(x)| \ge |x|^\beta$ with $0 < \beta < 1$, then $\left|\frac{f(x)-f(0)}{x-0}\right| \ge |x|^{\beta-1}$ tends to infinity as $x \to 0$, so the limit does not exist.

Problem 2. Suppose that $f^{(n)}(a)$ and $g^{(n)}(a)$ exist. Prove *Leibniz's formula*:

$$(f \cdot g)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) \cdot g^{(n-k)}(a).$$

Solution. The proof is by induction.

Base case $n = 0$: This is vacuous: $f \cdot g(a) = f(a)g(a)$.

Inductive step: Suppose the conclusion holds for some $n \geq 0$. Write

$$\begin{aligned} \frac{d^{n+1}}{dx^{n+1}}(f \cdot g)\bigg|_{x=a} &= \frac{d}{dx}\left(\frac{d^n}{dx^n}(f \cdot g)\right)\bigg|_{x=a} \\ &= \frac{d}{dx}\left(\sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) \cdot g^{(n-k)}(x)\right)\Bigg|_{x=a}. \end{aligned}$$

By linearity of the derivative, then the product rule

$$\begin{aligned} \frac{d^{n+1}}{dx^{n+1}}(f \cdot g)\bigg|_{x=a} &= \sum_{k=0}^{n} \binom{n}{k} \frac{d}{dx}\left(f^{(k)}(x) \cdot g^{(n-k)}(x)\right)\bigg|_{x=a} \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k+1)}(x)g^{(n-k)}(x) + f^{(k)}(x)g^{(n+1-k)}(x)\right)\bigg|_{x=a} \\ &= f^{(n+1)}(a) + g^{(n+1)}(a) \\ &\quad + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k}\right) f^{(k)}(a)g^{(n+1-k)}(a) \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(a)g^{(n+1-k)}(a). \end{aligned}$$

This completes the inductive step.

Problem 3.

a. (2 points) State the weighted (non-integral) Jensen's inequality.

b. (3 points) Using Jensen's inequality, or otherwise, prove the following *Power Mean Inequality.* Let $x_1, x_2, ..., x_n \in \mathbb{R}_{>0}$. If $0 < a < b$ then

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leq \left(\frac{1}{n}\sum_{i=1}^{n} x_i^a\right)^{\frac{1}{a}} \leq \left(\frac{1}{n}\sum_{i=1}^{n} x_i^b\right)^{\frac{1}{b}}.$$

(Hint: set $y_i = x_i^a$ to reduce to the case $a = 1$.)

Solution.

a. Let f be convex on an interval $[a, b]$. Let $x_1, x_2, ..., x_n \in [a, b]$ and let $0 < w_1, w_2, ..., w_n$ be some positive weights with $w_1 + ... + w_n = 1$. We have

$$f(w_1x_1 + ... + w_nx_n) \leq w_1f(x_1) + ... + w_nf(x_n).$$

b. Note that $x \mapsto x^a = \exp(a \log x)$ is increasing on $(0, \infty)$. Thus it is equivalent to prove

$$\left(\prod_{i=1}^{n} y_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} y_i \leq \left(\frac{1}{n}\sum_{i=1}^{n} y_i^{\frac{b}{a}}\right)^{\frac{a}{b}}$$

which is the result of raising each term in the inequalities to that ath power. The first inequality is the AM-GM inequality, which was verified in Lecture. Set $r = \frac{b}{a} > 1$. Then $f(x) = x^r$ satisfies

$$f''(x) = r(r-1)x^{r-2} > 0$$

for all $x > 0$ so f is convex on $(0, \infty)$. By Jensen's inequality with all weights $\frac{1}{n}$,

$$\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)^{r} \leq \frac{1}{n}\sum_{i=1}^{n} y_i^r$$

which is equivalent to the second inequality, after raising both sides to the $1/r$ power.

Problem 4. Use integration by parts to derive the recursion formula for $n \neq 0$,

$$\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx.$$

Solution. Set

$$u = \cos^{n-1} x, \qquad dv = \cos x dx$$

so that

$$du = -(n-1)\cos^{n-2} x \sin x dx, \qquad v = \sin x.$$

Integrate by parts to find

$$\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x dx$$

Use $\sin^2 x = 1 - \cos^2 x$ to rewrite the RHS as

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx$$

or

$$n \int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx.$$

Dividing both sides by n gives the claim.

Problem 5.

a. (3 points) Calculate $\lim_{n\to\infty}(\sqrt{n^2+n}-n)$.
b. (2 points) Evaluate $\lim_{x\to 1} x^{1/(1-x)}$.

Solution.

a. Use $a^2-b^2=(a+b)(a-b)$ to write

$$\sqrt{n^2+n}-n=\frac{n}{\sqrt{n^2+n}+n}=\frac{1}{\sqrt{1+\frac{1}{n}}+1}.$$

Since $\sqrt{x}$ is continuous at 1, it follows that

$$\lim_{n\to\infty}\sqrt{n^2+n}-n=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}=\frac{1}{2}.$$

b. The answer is $\frac{1}{e}$. To check this, write

$$\log x^{1/(1-x)}=\frac{\log x}{1-x}=\frac{\log(1-y)}{y}.$$

where $y=1-x$. Write $\log(1-y)=-y+o(y)$, so $\lim_{y\to 0}\frac{\log(1-y)}{y}=-1$, whence $\lim_{x\to 1}x^{1/(1-x)}=e^{-1}$.

Problem 6. Prove that, of all rectangles of a given perimeter, the square has the largest area.

Solution. Let the rectangle have perimeter p with length ℓ and width w. Thus $2(w+\ell) = p$. The area is given by $A(\ell) = \ell w = \ell\left(\frac{p}{2} - \ell\right)$. The problem reduces to maximize $A(\ell)$ subject to $0 \leq \ell \leq \frac{p}{2}$. At both endpoints, the area is 0. Calculate $A'(\ell) = \frac{p}{2} - 2\ell$. The only critical point is $\ell = \frac{p}{4}$, which must give the maximum. The maximum area is thus $\frac{p^2}{16}$, which is achieved by a square of side length $\frac{p}{4}$.