MATH 141, FALL 2016 MIDTERM 1

SEPTEMBER 28

Solve 4 of 6 problems. You may quote results stated during lecture, so long as you represent the result accurately.

Problem 1. Prove by induction

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

Solution. Base case ($n = 0$): The sum is empty, and hence $0 = \frac{0}{1}$ is true.

Inductive step: Assume for some $n \geq 0$ that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$. Write $\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$. Hence, by the recursive definition of the sum notation, then the inductive assumption,

$$\begin{aligned} \sum_{i=1}^{n+1} \frac{1}{i(i+1)} &= \frac{1}{(n+1)(n+2)} + \sum_{i=1}^{n} \frac{1}{i(i+1)} \\ &= \frac{1}{n+1} - \frac{1}{n+2} + \frac{n}{n+1} \\ &= \frac{n+1}{n+2}. \end{aligned}$$

This completes the inductive step.

Problem 2. Prove the following statements from the field axioms. For all a, $a \cdot 0 = 0 \cdot a = 0$. Also, 0 has no reciprocal.

Solution. Let 0 denote the additive identity and 1 denote the multiplicative identity. Write $1 = 1+0$ and $a = a \cdot 1 = a \cdot (1+0)$, then apply the distributive property to obtain $a = a \cdot 1 + a \cdot 0 = a + a \cdot 0$. Add the additive inverse of a on the left on both sides, and use associativity of addition to obtain

$$0 = (-a + a) = (-a + a) + a \cdot 0 = 0 + a \cdot 0 = a \cdot 0.$$

$0 \cdot a = 0$ follows by commutativity of multiplication. Suppose b is a reciprocal for 0. Then $0 \cdot b = 1$, but $0 \cdot b = 0$ and $0 \neq 1$, a contradiction.

Problem 3. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$ where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Solution. Since A is non-empty and bounded below, $a = \inf A$ exists. Since a is a lower bound for A, for all $x \in A$, $x \geq a$. Thus, for all $y \in -A$, $y = -x$ for some $x \in A$, whence $y = -x \leq -a$. This shows that $-a$ is an upper bound for $-A$.

To check that $-a$ is the least upper bound for $-A$, let $b \leq -a$ be an upper bound for $-A$. It follows that $-b \geq a$, and for all $x \in A$, $-x \in -A$, whence $-x \leq b$, so $-b \leq x$, so $-b$ is a lower bound for A. By definition of the inf, $-b \leq a$, so $-b = a$ and thus $-a = b$. This proves that $-a = \sup(-A)$.

Problem 4. Define $\sin^{-1} : [0, 1] \to [0, \frac{\pi}{2}]$ to be the inverse function of $\sin x$. Justify that $\sin^{-1}$ is integrable and calculate

$$\int_0^1 \sin^{-1}(t)dt.$$

Solution. As discussed in lecture, $\sin(x)$ is continuous and increasing on $\left[0, \frac{\pi}{2}\right]$, with $\sin 0 = 0$ and $\sin \frac{\pi}{2} = 1$. It follows that $\sin^{-1}$ is continuous and increasing as a function $[0, 1] \to [0, \frac{\pi}{2}]$, hence is integrable. Let $R = \left[0, \frac{\pi}{2}\right] \times [0, 1]$. As discussed in lecture,

$$\text{area}(R) = \int_0^{\frac{\pi}{2}} \sin(x)dx + \int_0^1 \sin^{-1}(x)dx,$$

so $\int_0^1 \sin^{-1}(x)dx = \frac{\pi}{2} - 1$.

Problem 5. State carefully the definition of a function continuous at a point p. Then prove that the function $f(x) = \frac{1}{x}$ is a continuous bijection from $(0,1)$ to $(1,\infty)$.

Solution. Function f is continuous at a point p if

- f is defined at p
- $\lim_{x\to p} f(x)$ exists and is equal to $f(p)$.

As verified in homework, in an ordered field, if $0 < a < b$ then $0 < \frac{1}{b} < \frac{1}{a}$. This proves that $f(x) = \frac{1}{x}$ is strictly decreasing as a function on $(0,1)$, and hence maps $(0,1) \to (1,\infty)$, and is injective. Given $y > 1$, $x = \frac{1}{y}$ satisfies $y = \frac{1}{x}$, and $0 < x < 1$. Thus f is surjective, and so bijective.

Let $0 < p < 1$. To prove $f(x) = \frac{1}{x}$ is continuous at p, given $\epsilon > 0$ choose $\delta = \min\left(\frac{p}{2}, \frac{2\epsilon}{p^2}\right)$. For $x \in (0,1)$ and $|x-p| < \delta$ one has $x > \frac{p}{2}$. It holds

$$\left|\frac{1}{x} - \frac{1}{p}\right| = \left|\frac{p-x}{xp}\right| < \frac{2|p-x|}{p^2} < \frac{p^2\delta}{2} \leq \epsilon,$$

proving the continuity.

Problem 6. State the pigeonhole principle. Using it, prove that among 11 numbers in the range 1 to 100, two differ by at most 9.

Solution. The pigeonhole principle: Let $0 < m < n$ be natural numbers, and let $[m]$ and $[n]$ denote the standard sets of cardinality m and n. There does not exist an injective map $[n] \to [m]$.

Divide the range [100] into 10 equally sized sets, $S_i = \{10(i-1) + j : 1 \leq j \leq 10\}$, $i = 1, 2, ..., 9$. Define $f : [11] \to [10]$ by $f(i)$ is the index of the set to which the ith number belongs. By the pigeonhole principle, f is not injective, and hence there exists $1 \leq i_1 < i_2 \leq 11$ for which $f(i_1) = f(i_2) = j$, say. Since the largest difference between two numbers in S_j is 9, the claim follows.