MAT 132

Midterm II Solutions.

This is a closed notes/ closed book/ electronics off exam.

Please write legibly and show your work.

Each problem is worth 20 points.

Full Name:						
Problem	1	2	3	4	5	Total
Grade						

Problem 1. The ice cream in an ice cream cone makes up a right circular cone of diameter 4 inches and height 5 inches, together with a spherical cap which extends to height 1 inch above the top of the cone. Find the volume of ice cream.

Solution 1. The volume of the ice cream in the cone is

$$\frac{1}{3}(\text{base}) \times (\text{height}) = \frac{20\pi}{3}.$$

To determine the radius r of the sphere making the spherical cap, form a right triangle with legs $r-1$, 2 and hypotenuse r by dropping a perpendicular from the center of the sphere to the center of the base of the cone, and then connecting it to a point on the boundary. Hence

$$r^2 - (r-1)^2 = 2r - 1 = 4,$$

so $r = \frac{5}{2}$. By the washer method, the volume of the spherical cap is

$$\int_{\frac{3}{2}}^{\frac{5}{2}} \pi \left(\left(\frac{5}{2}\right)^2 - x^2 \right) dx = \pi \left[\frac{25}{4}x - \frac{x^3}{3} \right]_{\frac{3}{2}}^{\frac{5}{2}} = \frac{13\pi}{6}.$$

Hence the total volume of ice cream is $\frac{20\pi}{3} + \frac{13\pi}{6} = \frac{53\pi}{6}$.

Problem 2. Find the center of mass of the region

$$R = \left\{(x, y) : x \geq 0, 0 \leq y \leq x\left(4 - x^2\right)\right\}.$$

Solution 2. The total area is

$$A = \int_0^2 4x - x^3 dx = \left[2x^2 - \frac{x^4}{4}\right]_0^2 = 4.$$

The moment in the x direction is

$$M_x = \int_0^2 x(4x - x^3)dx = \int_0^2 4x^2 - x^4 dx = \left[\frac{4x^3}{3} - \frac{x^5}{x}\right]_0^2 = \frac{64}{15}.$$

The moment in the y direction is

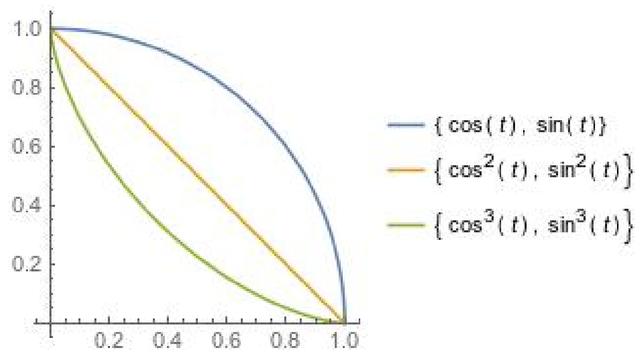
$$\begin{aligned} M_y = \int_0^2 \frac{1}{2}\left(4x - x^3\right)^2 dx &= \frac{1}{2}\int_0^2 x^6 - 8x^4 + 16x^2 dx \\ &= \frac{1}{2}\left[\frac{x^7}{7} - \frac{8x^5}{5} + \frac{16x^3}{3}\right]_0^2 \\ &= \frac{512}{105}. \end{aligned}$$

Thus $(\overline{x}, \overline{y}) = \left(\frac{16}{15}, \frac{128}{105}\right)$.

Problem 3.

a. Sketch the three curves

$$(\cos\theta, \sin\theta), \quad (\cos^2\theta, \sin^2\theta), \quad (\cos^3\theta, \sin^3\theta), \qquad 0 \le \theta \le \frac{\pi}{2}$$



b. Find the length of the astroid

$$A = \left(\cos^3\theta, \sin^3\theta\right), \qquad 0 \le \theta \le 2\pi.$$

Solution 3.

b. Since $x'(\theta) = -3\cos^2\theta\sin\theta$, $y'(\theta) = 3\sin^2\theta\cos\theta$,

$$\begin{aligned} x'(\theta)^2 + y'(\theta)^2 &= 9\cos^4\theta\sin^2\theta + 9\cos^2\theta\sin^4\theta \\ &= 9\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta) \\ &= \frac{9}{4}(\sin 2\theta)^2. \end{aligned}$$

Thus the arc length is equal to

$$L = \int_0^{2\pi} \sqrt{x'(\theta)^2 + y'(\theta)^2} d\theta = \frac{3}{2}\int_0^{2\pi} |\sin 2\theta| d\theta = 6.$$

Problem 4.

a. Find the work done by gravity when a 50 pound bucket of water is pulled up a 20 foot well by a rope weighing one pound per foot.

b. Now suppose that the bucket is pulled upward at a constant rate of .5 foot per second and that water leaks out of the bucket at a rate of 1 pound per second, and that there is still water in the bucket when it reaches the top of the well. Find the work done by gravity in this case.

Solution 4.

a. When the bucket has been lifted distance x, the weight of the bucket and rope is $70 - x$. Thus the work done by gravity is

$$-\int_0^{20} 70 - x dx = -\left[70x - \frac{x^2}{2}\right]_0^{20} = -1200\text{ft-lb.}$$

b. At time t seconds, the bucket has been raised $t/2$ feet, and t pounds of water has leaked out. Thus, when the bucket has been raised x feet, $2x$ pounds of water have leaked out, so that the total weight of rope and bucket is $70 - 3x$. The work done now is

$$-\int_0^{20} 70 - 3x dx = -\left[70x - \frac{3x^2}{2}\right]_0^{20} = -800\text{ft-lb.}$$

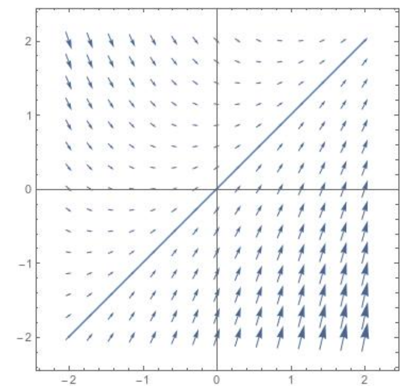
Problem 5.

a. Match each differential equation to the corresponding vector field, and sketch the solution with initial value $y(0) = 0$.

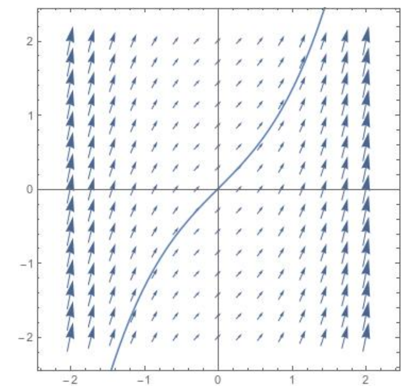
   i. $y' = x^2 + y^2 - 1$ (C)

   ii. $y' = 1 + x - y$ (A)

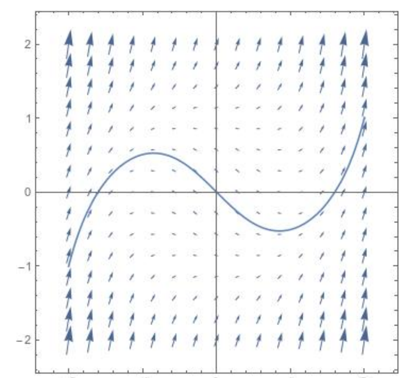
   iii. $y' = 1 + x^2$ (B)



A $y' = 1 + x - y$



B $y' = 1 + x^2$



C $y' = x^2 + y^2 - 1$

b. Use Euler's method with step $h = \frac{1}{3}$ to estimate $y(1)$ given the initial value problem

$$y' = 9(x^2 + y^2), \qquad y(0) = 0.$$

Does Euler's method give an over or an underestimate? Why?

Solution 5.

b. The output of Euler's method with step $\frac{1}{3}$ is shown, the estimated value is 2.

x	y	y'
0	0	0
$\frac{1}{3}$	0	1
$\frac{2}{3}$	$\frac{1}{3}$	5
1	2	

Since $y'' = 18x + 18yy' = 18x + 162y(x^2 + y^2) \geq 0$ along the solution curves in the first quadrant, the solution curves are convex, so the secant lines lie below the curves. Thus Euler's method gives an underestimate.