Sandpile Dynamics on Tiling Graphs

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History of Sandpiles

- In a 1987 paper by Bak, Tang and Wiesenfeld.
- A Google Scholar search returns >1800 hits.

Sandpile Dynamics of $G = (V, E, S)$

- $\sigma(v)$: # of chips distributed on each vertex $v \in V$.
- **Stable** vertex v: $\sigma(v) < \deg(v)$.
- **Toppling**: sending out a chip to each neighboring vertex from one unstable vertex.
- Passed chips to $s \in S$ are removed.
- In each step a grain of sand is dropped on a uniform vertex and all legal topplings occur.

Graph Laplacian Δ

- $\Delta f(v) = \deg(v) f(v) - \sum_{(v,w)\in E} f(w)$.
- Δ': a graph Laplacian obtained by omitting the row and column corresponding to the sink.

Sandpile Group of a Graph

- The sandpile group of $G = (V, E, S)$ is isomorphic to $\mathcal{G} = \mathbb{Z}^{V\setminus\{s\}}/\Delta'\mathbb{Z}^{V\setminus\{s\}}$.
- The dual group is isomorphic to $\hat{\mathcal{G}} = (\Delta')^{-1}\mathbb{Z}^{V\setminus\{s\}}/\mathbb{Z}^{V\setminus\{s\}}$.

Functions Harmonic Modulo 1

- ξ is **harmonic modulo 1** if $\Delta\xi \equiv 0 \bmod 1$.
- $f(\xi) = \sum_{x\in\mathscr{T}} 1 - \cos(2\pi\xi_x)$

The Fourier Coefficients

- The eigenvalue of the Markov chain associated to $\xi \in \hat{\mathcal{G}}$ is the **Fourier coefficient** of the measure μ driving the random walk at frequency ξ:

$$\hat{\mu}(\xi) = \frac{1}{|V|}\left(1 + \sum_{v\in V\setminus\{s\}} e(\xi_v)\right).$$

Sandpiles with Periodic and Open Boundary

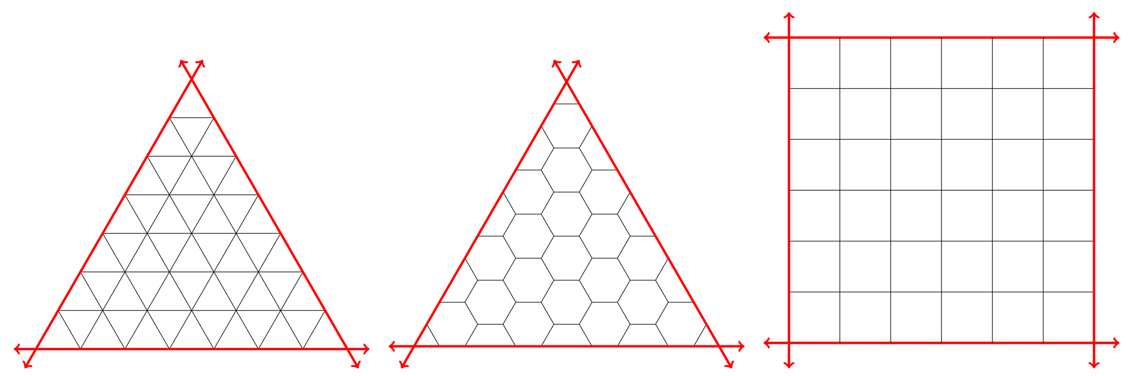

Figure 1: The triangular, honeycomb, and square lattice configurations with open boundary condition.

The Measure Driving Sandpile Dynamics

- **Group Convolution**: A random walk driven by a probability measure μ on a group has distribution at step n given by μ^{*n} where $\mu^{*1} = \mu$ and $\mu^{*n} = \mu * \mu^{*(n-1)}$.
- **Total Variation Mixing Time**: Let a measure μ be driving sandpile dynamics on the group of recurrent sandpile states $\mathcal{G}(G)$ with uniform measure $\mathbb{U}_{\mathcal{G}(G)}$.

$$t_{\text{mix}} = \min\left\{k : \left\|\mu^{*k} - \mathbb{U}_{\mathcal{G}(G)}\right\|_{\text{TV}(\mathcal{G}(G))} < \frac{1}{e}\right\}.$$

- Given a sequence of graphs G_n the sandpile dynamics is said to satisfy the **cut-off phenomenon** in total variation if, for each $\epsilon > 0$,

$$\left\|\mu^{*\lceil(1-\epsilon)t_{\text{mix}}\rceil} - \mathbb{U}_{\mathcal{G}(G_n)}\right\|_{\text{TV}(\mathcal{G}(G_n))} \to 1,$$
$$\left\|\mu^{*\lceil(1+\epsilon)t_{\text{mix}}\rceil} - \mathbb{U}_{\mathcal{G}(G_n)}\right\|_{\text{TV}(\mathcal{G}(G_n))} \to 0$$

as $n \to \infty$.

Spectral Gap and Spectral Factors

- $\mathfrak{S}_S$: the group generated by reflections in a set S of co-dimension 1 subspaces.
- $\mathscr{H}_S(\mathscr{T})$: a set of harmonic modulo 1 functions f on $\mathscr{T}$ which are anti-symmetric in each plane of S, identified with functions on $\mathscr{T}/\mathfrak{S}_S$.
- γ_i (**Spectral Parameters**): For $0 \leq i \leq d$,

$$\gamma_i = \inf_{|S|=i} \inf_{\substack{\xi\in\mathscr{H}_S(\mathscr{T})\\ \xi\not\equiv 0 \bmod 1}} \sum_{x\in\mathscr{T}/\mathfrak{S}_S} 1 - \cos(2\pi\xi_x).$$

- Γ_j (**jth Spectral Factor**): For dimension $d \geq 2$, $\Gamma_j = \frac{d}{j}\gamma_j$ and $\Gamma = \max_j \Gamma_j$.

Determination of Spectral Gap and Spectral Factors

- Let $\Delta\xi = \nu$, $\|\xi\|_\infty \leq \frac{1}{2}$.
- Given a set $S \subset \mathscr{T}$, a lower bound for $f(\xi)$ is obtained as the solution of the **optimization programs** $Q(S,\nu)$ and $P(S,\nu)$.
- $Q(S,\nu)$: minimizes $\sum_{d(w,S)\leq 1} 1 - c(x_w)$ where $\forall u \in S$, $(\deg u)x_u - \sum_{d(w,u)=1} x_w = \nu_u$ and $-\frac{1}{2} \leq x_w \leq \frac{1}{2}$.
- $P(S,\nu)$: a relaxed optimization program with $\forall u \in S$, $(\deg u)x_u + \sum_{d(w,u)=1} x_w \geq \nu_u$.
- P and Q are monotone in S, and the value at the union of two sets is additive if the variables are disjoint.
- A connected component analysis is performed to identify the extremal ν.
- A method is given to obtain the Fourier transform of the Green's function of an arbitrary tiling, and ξ is computed as a Fourier integral.

Important Theorems

- For planar periodic tilings satisfying a reflection condition, the asymptotic mixing time is equal for periodic and open boundary conditions.
- For the D4 lattice in dimension 4, there is a choice of boundary with the open boundary mixing controlled by the 3 dimensional boundary.
- With an open boundary condition we prove cut-off in total variation mixing at a time proportional to the spectral factor, which combines a spectral and dimension component.

Extremal Configurations

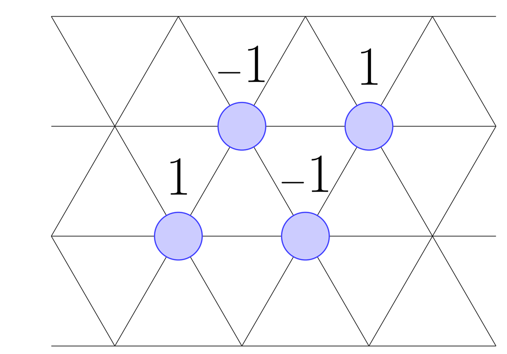

Figure 2: The extremal configuration for the triangular lattice.

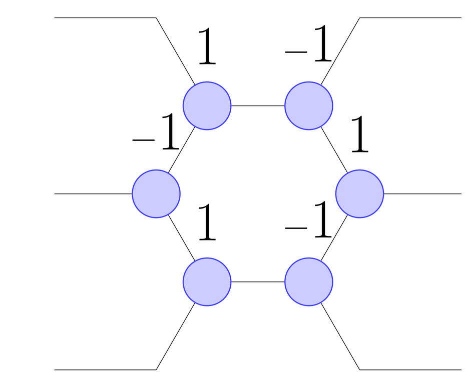

Figure 3: The extremal configuration for the honeycomb lattice.

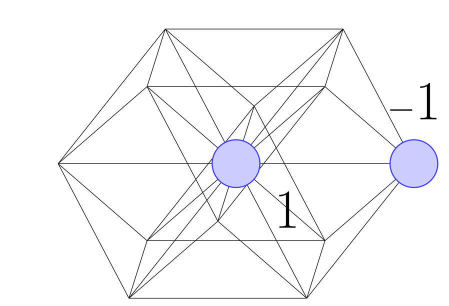

Figure 4: The extremal configuration for the face centered cubic lattice.

Proof of cut-off

- A clustering is performed on $\nu = \Delta'\xi$.
- Using van der Corput's inequality, separated clusters make independent contributions to $\hat{\mu}(\xi)$.
- The proof of cut-off modifies the spectral proof for simple random walk on $(\mathbb{Z}/2\mathbb{Z})^N$.

References

[1] Robert Hough and Hyojeong Son.
Sandpile dynamics on periodic tiling graphs.

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