MAT 542 Complex Analysis I

Problem Set 9

due Tuesday, April 24

Problem 1. Consider a function $f(z)$ in a neighborhood of i given by a branch of $\sqrt{z(1-z)}$, so that

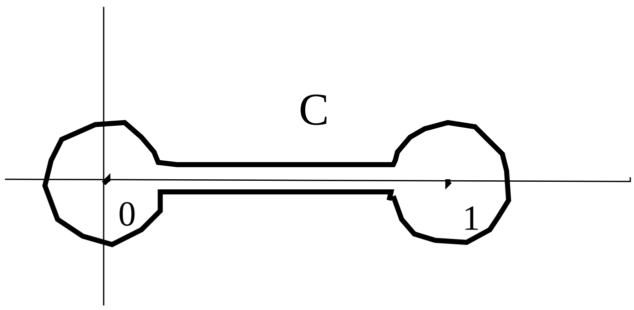
$$f(z) = \exp\left(\frac{1}{2}\log\left(z(1-z)\right)\right),$$

where log stands for the principal value of logarithm.

Show that $f(z)$ can be analytically continued along any curve $\gamma \subset \mathbb{C} - \{0,1\}$, and explain what happens when you continue f along various closed curves going around 0 and/or 1. (You don't have to analyze all possible curves; just illustrate the situation by a few examples.)

Show that a branch of $\sqrt{z(1-z)}$ is well-defined on $\mathbb{C}-[0,1]$, even though $\mathbb{C}-[0,1]$ is not simply connected. (By a branch here we mean a direct analytic continuation of f into the whole $\mathbb{C}-[0,1]$).

Let f denote such a branch. For the path C shown below, compute the integral $\displaystyle\int_C \frac{1}{\sqrt{z(1-z)}}\,dz$ via the residue at infinity. Use this to compute $\displaystyle\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx$.



Compute the (improper) integral $\displaystyle\int_0^1 \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx$ by calculus methods to check your answer.

Problems 2-5 use Schwarz reflection which we'll discuss on Thursday.

Problem 2. Let C be an arc of the unit circle $\{z : |z| = 1\}$, and let U be an open set inside the circle, having that arc as a piece of its boundary. Let f be an analytic function on U that maps U into the upper half-plane. If f is continuous on C and takes real values on C, prove that f can be continued across C by the relation

$$f(z) = \overline{f(1/\bar{z})}.$$

Problem 3. Let U, C, and f be as in Problem 2 but suppose that instead of taking real values on C, f takes on values on the unit circle, i.e. $|f(z)| = 1$ for $z \in C$. Show that the analytic continuation of f across C is now given by

$$f(z) = 1/\overline{f(1/\bar{z})}.$$

Problem 4. Let f be a function which is continuous on the closed unit disk and analytic on the open disk. Assume that $|f(z)| = 1$ whenever $|z| = 1$. Show that f can be extended to a meromorphic function with at most a finite number of poles in the whole complex plane.

Problem 5. Let f be a meromorphic function on the open unit disk and assume that f has a continuous extension to the boundary circle. Assume also that f has only a finite number of poles in the unit disk, and that $|f(z)| = 1$ whenever $|z| = 1$. Prove that f is a rational function.

Problem 6. Let $P(z, w)$ be a polynomial in two complex variables. Let f_0 be an analytic function on A_0 and g_0 an analytic function on B_0. Suppose γ is a curve and pairs (f_0, A_0) and (g_0, B_0) can be analytically continued along γ to pairs (f_n, A_n) and (g_m, B_m). If $P(f_0, g_0) = 0$ everywhere in $A_0 \cap B_0$, show that $P(f_n, g_m) = 0$ everywhere in $A_n \cap B_m$. (That is, the relation $P(\cdot, \cdot) = 0$ persists under analytic continuation.)