MAT 542 Complex Analysis I

Problem Set 8

due Tuesday, April 17

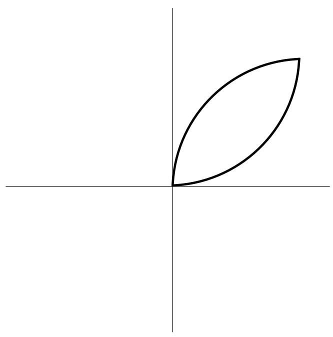
Problem 1. We saw that the function $z + 1/z$ establishes a conformal isomorphism between the set $V = \{z \in \mathbb{C} : |z| > 1, \operatorname{Im} z > 0\}$ and the upper half-plane H.

Use Möbius transformations to find a (possibly different) conformal isomorpism between these two sets: first, find an element of $\operatorname{Aut}(\bar{\mathbb{C}})$ which maps the upper half-disk $V = \{z : |z| < 1, \operatorname{Im} z > 0\}$ onto the first quadrant $\{z : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$. Combine it with elementary maps to obtain a conformal isomorphism $V \longrightarrow H$.

Problem 2. Let $U \subset \mathbb{C}$ be the region

$$U = \{z : |z - 1| < 1 \text{ and } |z - i| < 1\}.$$

(See the picture.)



Find an explicit conformal isomorphism between U and the unit disk $D = \{|z| < 1\}$. (Use Möbius transformations.)

Problem 3. Determine the automorphism group of the doubly-punctured plane $\mathbb{C} \setminus \{0, 1\}$. (Be careful: you can't assume that all automorphisms extend to $\bar{\mathbb{C}}$.)

Problem 4. Let $U \subset \mathbb{C}$ be open, connected. Suppose that a sequence of holomorphic functions $\{f_j\}$ on U converges to a holomorphic function f uniformly on every compact set in U. Suppose there is an open set V such that $f_j(U) \subset V$ for all j. Show that either $f(U) \subset V$ or f is a constant function taking its value on ∂V.

Problem 5. Let Φ be the family of all analytic functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

on the open unit disk, such that $|a_n| \leq n$ for each n. Show that Φ is relatively compact.

Problem 6. Let $U \subset \mathbb{C}$ be an open set. The set $\operatorname{Hol}(U)$ of holomorphic functions on U can be endowed with a metric, as follows.

Let $\{K_j\}$ $(j = 1, 2, \dots)$ be a sequence of compact sets such that $K_j \subset \text{interior}(K_{j+1})$ and $\bigcup_j K_j = U$. For $f, g \in \text{Hol}(U)$, set

$$\rho_j(f, g) = \min(1, \sup_{z \in K_j} |f(z) - g(z)|).$$

Define

$$\rho(f, g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \rho_j(f, g).$$

(i) Check that ρ is well-defined, and $(\text{Hol}(U), \rho)$ is indeed a metric space.

(ii) Show that a sequence $\{f_n\}$ in $\text{Hol}(U)$ converges uniformly on every compact subset of U if and only if it converges with respect to the metric ρ.

(iii) Show that a family of functions $\Phi \subset \text{Hol}(U)$ is relatively compact if and only if it is relatively compact in the metric spaces sense, i.e. its closure in $(\text{Hol}(U), \rho)$ is compact.