Problem 1. We saw that the function \( z + 1/z \) establishes a conformal isomorphism between the set \( V = \{ z \in \mathbb{C} : |z| > 1, \text{Im} z > 0 \} \) and the upper half-plane \( H \).

Use Möbius transformations to find a (possibly different) conformal isomorphism between these two sets: first, find an element of \( \text{Aut}(\mathbb{C}) \) which maps the upper half-disk \( V = \{ z : |z| < 1, \text{Im} z > 0 \} \) onto the first quadrant \( \{ z : \text{Re} z > 0, \text{Im} z > 0 \} \). Combine it with elementary maps to obtain a conformal isomorphism \( V \rightarrow H \).

Problem 2. Let \( U \subset \mathbb{C} \) be the region

\[
U = \{ z : |z - 1| < 1 \text{ and } |z - i| < 1 \}.
\]

(See the picture.)

Find an explicit conformal isomorphism between \( U \) and the unit disk \( D = \{ |z| < 1 \} \).

(Use Möbius transformations.)

Problem 3. Determine the automorphism group of the doubly-punctured plane \( \mathbb{C} \setminus \{ 0, 1 \} \).

(Be careful: you can’t assume that all automorphisms extend to \( \mathbb{C} \).)

Problem 4. Let \( U \subset \mathbb{C} \) be open, connected. Suppose that a sequence of holomorphic functions \( \{ f_j \} \) on \( U \) converges to a holomorphic function \( f \) uniformly on every compact set in \( U \). Suppose there is an open set \( V \) such that \( f_j(U) \subset V \) for all \( j \). Show that either \( f(U) \subset V \) or \( f \) is a constant function taking its value on \( \partial V \).

Problem 5. Let \( \Phi \) be the family of all analytic functions

\[
f(z) = z + a_2 z^2 + a_3 z^3 + \ldots
\]
on the open unit disk, such that \( |a_n| \leq n \) for each \( n \). Show that \( \Phi \) is relatively compact.

Problem 6. Let \( U \subset \mathbb{C} \) be an open set. The set \( \text{Hol}(U) \) of holomorphic functions on \( U \) can be endowed with a metric, as follows.
Let \( \{K_j\} \) \((j = 1, 2, \ldots)\) be a sequence of compact sets such that \( K_j \subset \text{interior}(K_{j+1}) \) and \( \bigcup_j K_j = U \). For \( f, g \in \text{Hol}(U) \), set
\[
\rho_j(f, g) = \min(1, \sup_{z \in K_j} |f(z) - g(z)|).
\]
Define
\[
\rho(f, g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \rho_j(f, g).
\]

(i) Check that \( \rho \) is well-defined, and \((\text{Hol}(U), \rho)\) is indeed a metric space.

(ii) Show that a sequence \( \{f_n\} \) in \( \text{Hol}(U) \) converges uniformly on every compact subset of \( U \) if and only if it converges with respect to the metric \( \rho \).

(iii) Show that a family of functions \( \Phi \subset \text{Hol}(U) \) is relatively compact if and only if it is relatively compact in the metric spaces sense, i.e. its closure in \((\text{Hol}(U), \rho)\) is compact.